MIDDLE NUCLEUS = CENTER
IN SEMIPRIME JORDAN ALGEBRAS

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ABSTRACT. A. A. Albert showed that the middle nucleus and center coincide for a simple Jordan algebra finite-dimensional over a field of characteristic $\not= 2$. E. Kleinfeld extended this to arbitrary simple Jordan algebras of characteristic $\not= 2$. Recently this result has played a crucial role in the structure theory of E. Zelmanov. In this note we extend the result to linear Jordan algebras with no derivation-invariant trivial ideals.

The left, middle, and right nucleus $N_l(A)$, $N_m(A)$, $N_r(A)$ of any nonassociative algebra $A$ consists of the elements $n \in A$ with $[n,A,A] = 0$, $[A,n,A] = 0$, $[A,A,n] = 0$ respectively, where the associator is given by $[x,y,z] = (xy)z - x(yz)$. The nucleus $N(A)$ consists of the elements in all three nuclei simultaneously, and the center $C(A)$ consists of the nuclear elements which commute with $A$, $[n,A] = 0$. A derivation of $A$ is a linear transformation $D$ satisfying

$$D(xy) = D(x)y + xD(y)$$

and hence necessarily

$$D([x,y,z]) = [D(x),y,z] + [x,D(y),z] + [x,y,D(z)]$$

as well. From this we see each nucleus is derivation-invariant, $D(N_l(A)) \subseteq N_l(A)$ for all $D$ (the same is true of the center). When $A$ is commutative we have

1. $[x,y,z] = -[z,y,x]$,
2. $[x,y,z] + [y,z,x] + [z,x,y] = 0,$

so $C(A) = N(A) = N_l(A) = N_r(A) \subseteq N_m(A)$. Our goal is to show conversely that for semiprime Jordan algebras the reverse inclusion holds as well. This has become important in Zelmanov's work [4], where expressions $[x_3, [x_1,x_2]^2,x_4]$ play a crucial role and one wants to know that if these vanish for all $x_3, x_4$ (i.e. $[x_1,x_2]^2$ is middle-nuclear) then $[x_1,x_2]^2$ is actually central.

We restrain our quadratic sympathies and work entirely with (nonunital) linear Jordan algebras $J$ over a ring of scalars $\Phi$ containing $\frac{1}{2}$. Thus $J$ has product $xy$ satisfying the Jordan axioms

3. $xy = yx$, i.e. $[x,y] = 0$,
4. $(x^2y)x = x^2(yx)$, i.e. $[x^2,y,x] = 0$.

In addition to the left multiplication operator $L_x(z) = xz$, the $U$-operator

5. $U(x) = (2L_x^2 - L_{x^2})(x)$

plays an important role even in the linear theory. It satisfies the identity

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The inner derivations $D_{x,y}$ are defined by
\[(9)\ D_{x,y}(z) = [L_x, L_y](z) = -[x, z, y];
\]
any derivation $D$ satisfies
\[(10)\ D(U_{x,y}) = U_{D(x), y} + U_{x, D(y)}, \quad [D, D_{x,y}] = D_{D(x), y} + D_{x, D(y)}.
\]
A Jordan algebra is semiprime if it contains no trivial ideals $B \ (UBB = 0)$ and is $D$-semiprime if it contains no nonzero trivial ideals $B$ invariant under all derivations.

**THEOREM.** The middle nucleus and center coincide, $N_m(J) = C(J)$, for any $D$-semiprime linear Jordan algebra $J$.

**PROOF.** We must show $n \in N_m(J) \Rightarrow n \in N(J)$, i.e. $[J, n, J] = 0 \Rightarrow [n, J, J] = 0$. Following Albert [1] we introduce an ideal
\[(11)\ B = [n, J, J]
\]
which measures the failure of $n$ to lie in $N(J)$. Here $B$ is spanned by all associators $[n, x, y]$ for $x, y \in J$, where
\[(12)\ [n, x, y] = [n, y, x]
\]
since $0 = [n, x, y] + [x, y, n] + [y, n, x] = [n, x, y] - [n, y, x] + 0$ by (4), (3), and the definition of $n \in N_m(J)$. Thus $[n, x, y] = \frac{1}{2}([n, x, y] + [n, y, x])$ results from linearizing $[n, x, x]$, so
\[(13)\ B \text{ is spanned by all } [n, x, x] \text{ for } x \in J.
\]
To see $B$ is actually an ideal, note $JB$ is spanned by all $y[n, x, x] = -x[n, y, x] + [n, xy, x]$ (by (1), (9)) = $-x[n, y, x] + [n, xy, x]$ (by (12)) = $-\frac{1}{2}[n, x^2, y] + [n, xy, x]$ (by (1)) $\in [n, J, J] = B$.

Now by (13), $[n, B, J]$ is spanned by elements $[n, [n, x, x], y] = -[[n, x, x], y, n] - [y, n, [n, x, x]]$ (by (4)) = $D_n D_{n,x}(y) + 0$ (by (9)) = $[[D_n, x], D_{n,x}] - D_{D_{n}, x}(n) x]}(y)$ (by (10)) = 0 (since $D_{J,J}(n) = 0$), so
\[(14)\ [n, B, J] = 0.
\]
Then $0 = [n, B, J] \supset [n, UB, J] = U[n, B, J] + UB[n, J, J]$ (by (9), (10)) = $0 + UB B$ implies $UB B = 0$. If $J$ is semiprime then $B = [n, J, J] = 0$ by (11) and $n \in N(J)$.

If $J$ is merely $D$-semiprime we must work a little harder. We claim $C = [N_m(J), J, J] = \sum[n, J, J]$ is a trivial $D$-invariant ideal. $C$ is an ideal since each $[n, J, J]$ is by (11), and it is $D$-invariant by (2) and the fact that $N_m(J)$ is $D$-invariant. As in (14) we have
\[(14')\ [n, C, J] = 0
\]
for any $n \in N_m(J)$ since by (12) this is spanned by all $[n, [n', x, x], y] = [[n, n', y], \ x, x] + [n', [n, x, y], x] + [n', x, [n, x, y]]$ (by (9), (2)) = $0 + 2[n', [n, x, y], x]$ (by (12) and $n' \in N_m(J)$) = $2[n', [n, x, y]] x = 2n' \{n, [x, y] x\}$ (by (12)) = $[n, x^2, y] - [n, n'y, x^2]$ (by (1), (9)) = 0 (by (12)). Then $0 = [n, C, J] \supset [n, CJ, J] = [n, C, J] J + C[n, J, J]$ (by (1)) = $0 + C[n, J, J]$ (by (14')); this holds for all $n \in N_m(J)$, so $0 = CC$ and $C$ is trivial as well as $D$-invariant. If $J$ is $D$-semiprime this forces $C = 0$, so $[N_m(J), J, J] = 0$ and $N_m(J) \subset N(J)$. □
It is easy to give examples to show that the semiprimeness hypothesis is needed. If we take \( J = F/K \) for \( F \) the free Jordan algebra on \( x, y, K \) the ideal generated by all \([a, y, b]\) for \( a, b \in F \), then \( n = y + K \) is middle-nuclear in \( J \) by construction but is not central: \([n, J, J] \neq 0 \) since \([y, x, x] \notin K \). \( K \) is graded, vanishing in total degrees < 3, and in degree \( x^2y^1 \) it is spanned by \([x, y, x] = 0 \), so \([y, x, x] \notin K \).

We can also easily give special examples \( J = A^+ \) for \( A \) an associative algebra. Here \( n \in N_m(J) \iff \{[n, J], J\} = 0 \) and \( n \in N_i(J) \iff \{[n, J, J] = 0 \). If \( A = \Phi E_{11} + \Phi E_{12} + \Phi E_{22} \) consists of all upper triangular \( 2 \times 2 \) matrices over \( \Phi \), then \([J, J] = \Phi E_{12} \), so \( n = E_{12} \) lies in \( N_m(J) \), yet \( n \notin N_i(J) \) since \([n, E_{11}], E_{11} = -[E_{12}, E_{11}] = E_{12} \neq 0 \). Note that \( n \) falls in the trivial ideal \( B = \Phi E_{12} \).

Another example is the algebra \( J(Q, c) \) determined by a quadratic form \( Q \); here \( n \in N_m(J) \iff \{[n, J, J] = 0 \). If \( A = E_1 + E_i2 + E_2^2 \) consists of all upper triangular \( 2 \times 2 \) matrices over \( E \), then \([J, J] = E_i2 \), so \( n = E_i2 \) lies in \( N_m(J) \), yet \( n \notin N_0(J) \) since \([n, E_i2], E_{i2} = -[E_i2, E_{i2}] = E_{i2} \neq 0 \). Note that \( n \) falls in the trivial ideal \( B = \Phi E_{12} \).

Without reference to middle nuclei we can establish

**Theorem.** If \( J \) is a semiprime linear Jordan algebra then \( C(I) \subset C(J) \) for any ideal \( I < J \). More generally, \( C(J) \subset C(J) \) as soon as \( J < J \) and \( J \) contains no trivial ideals of \( J \) invariant under all derivations of \( J \) which map \( J \) into itself.

**Proof.** The first assertion follows from the second, since by a result of Slin'ko [4] if \( J \) is semiprime so is any \( I < J \). Assume \( J \) is \( D \)-semiprime in \( J \) in the above sense; for convenience we may assume that \( J \) is unital. We must show that if \( c \in C(J) \) then \( c \in C(J) = N_t(J) \), i.e. \([c, J, J] = 0 \).

All the derived ideals \( J^{(n)} \) (where \( J^{(0)} = J, J^{(n+1)} = U_{J^{(n)}} J^{(n)} \) are invariant under the indicated derivations and remain ideals in \( J \), as are their annihilators \( J^{(n)} \) (if \( B < J \) is invariant so is \( B\perp = \{z \in J \{z, B, J\} = 0 \}, and \( B\perp < J \) since \( \{z, B, J\} \in J\{z, B, J\} \subset J\{z, B, J\} = 0 \)).

Moreover, \( B = J \cap J^{(n)} \) are solvable: \( B^{(n)} \subset J^{(n)} \cap J^{(n)} \perp, B^{(n+1)} = 0 \). If \( J \) contains no trivial invariant ideals then it contains no solvable invariant ideals of \( J \), so \( B = 0 \):

\[
\text{(15) if } J \text{ is } D \text{-semiprime in } J \text{ then } J \cap J^{(n)} \perp = 0.
\]

If \( D(J^{(n)}) = 0 \) then \( D(J) \subset J^{(n+1)} \perp \) since \( D(J), J^{(n+1)}, J \subset \{D(J^{(n)}), J^{(n)}\} \) (because \( \{d, U_{x,y}, \tilde{a}\} = \{d, x, \{y, x, \tilde{a}\} = \{d, U_{\tilde{x}, \tilde{a}}, y\} \) where \( x, y \in J^{(n)} < J \) = \( D(\{J, J^{(n)}\}, J^{(n)})\) = \( \{J, J^{(n)}\}, D(J^{(n)})\) = \( J, J^{(n)}\), \( D(J^{(n)})\) = 0 (because \( J^{(n)} < J \), \( D(J^{(n)}) = 0 \), so from (15) we see

\[
\text{(16) if } J \text{ is } D \text{-semiprime in } J \text{ then } D(J^{(n)}) = 0 \Rightarrow D(J) = 0
\]

for any derivation of \( J \) into \( J \).

In particular, for \( D = D_{c,J} \) as in (9) we see \([c, J, J] = 0 \) (by \( c \in N_i(J) \)) implies \([c, J, J] = 0 \), hence for \( D = D_{c,J} \) we see \([c, J^2, J] = -[J^2, J, c] = [J, c, J^2] \) (by (4)) \( \subset [c, J, J^2] + [J, c, J^2] + [J, J^2] \) (by (3) and linearized (6)) \( \subset [c, J, J] + [J, c, J] \) (by \( J < J \) = 0 (by the above and \( c \in N_m(J) \)) implies \([c, J, J] = 0 \) as desired. \( \square \)
REFERENCES


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