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**MIDDLE NUCLEUS = CENTER IN SEMIPRIME JORDAN ALGEBRAS**
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**ABSTRACT.** A. A. Albert showed that the middle nucleus and center coincide for a simple Jordan algebra finite-dimensional over a field of characteristic \( p \neq 2 \). E. Kleinfeld extended this to arbitrary simple Jordan algebras of characteristic \( p \neq 2 \). Recently this result has played a crucial role in the structure theory of E. Zelmanov. In this note we extend the result to linear Jordan algebras with no derivation-invariant trivial ideals.

The left, middle, and right nucleus \( N_l(A), N_m(A), N_r(A) \) of any nonassociative algebra \( A \) consists of the elements \( n \in A \) with \([n, A, A] = 0, [A, n, A] = 0, [A, A, n] = 0\) respectively, where the associator is given by \([x, y, z] = (xy)z - x(yz)\). The nucleus \( N(A) \) consists of the elements in all three nuclei simultaneously, and the center \( C(A) \) consists of the nuclear elements which commute with \( A \), \([n, A] = 0\). A derivation of \( A \) is a linear transformation \( D \) satisfying

\[
D(xy) = D(x)y + xD(y)
\]

and hence necessarily

\[
D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)]
\]

as well. From this we see each nucleus is derivation-invariant, \( D(N_l(A)) \subseteq N_l(A) \) for all \( D \) (the same is true for the center). When \( A \) is commutative we have

\[
\begin{align*}
(3) & \ [x, y, z] = -[z, y, x], \\
(4) & \ [x, y, z] + [y, z, x] + [z, x, y] = 0,
\end{align*}
\]

so \( C(A) = N(A) = N_l(A) = N_r(A) \subseteq N_m(A) \). Our goal is to show conversely that for semiprime Jordan algebras the reverse inclusion holds as well. This has become important in Zelmanov's work [4], where expressions \([x_3, [x_1, x_2]^2, x_4]\) play a crucial role and one wants to know that if these vanish for all \( x_3, x_4 \) (i.e. \([x_1, x_2]^2\) is middle-nuclear) then \([x_1, x_2]^2\) is actually central.

We restrain our quadratic sympathies and work entirely with (nonunital) linear Jordan algebras \( J \) over a ring of scalars \( \Phi \) containing \( \frac{1}{2} \). Thus \( J \) has product \( xy \) satisfying the Jordan axioms

\[
\begin{align*}
(5) & \ xy = yx, \text{ i.e. } [x, y] = 0, \\
(6) & \ (x^2y)x = x^2(yx), \text{ i.e. } [x^2, y, x] = 0.
\end{align*}
\]

In addition to the left multiplication operator \( L_x(z) = xz \), the \( U \)-operator

\[
U_x(z) = (2L_x^2 - L_{x^2})(z)
\]

plays an important role even in the linear theory. It satisfies the identity

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(8) \( L_y U_x + U_x L_y = U_{yx,x} \) \((U_{x,y} = U_{x+y} - U_x - U_y)\).

The inner derivations \( D_{x,y} \) are defined by

(9) \( D_{x,y}(z) = [L_x, L_y](z) = -[x, z, y] \);

any derivation \( D \) satisfies

(10) \( D(U_{xy}) = U_{D(x),y} + U_{x,D(y)}, [D, D_{x,y}] = D(D(x),y) + D_{x,D(y)}. \)

A Jordan algebra is semiprime if it contains no trivial ideals \( B (U_B B = 0) \) and is\

\( D\)-semiprime if it contains no nonzero trivial ideals \( B \) invariant under all derivations.

**Theorem.** The middle nucleus and center coincide, \( N_m(J) = C(J), \) for any
\( D\)-semiprime linear Jordan algebra \( J. \)

**Proof.** We must show \( n \in N_m(J) \Rightarrow n \in N_l(J), \) i.e. \([J, n, J] = 0 \Rightarrow [n, J, J] = 0. \) Following Albert \([1]\) we introduce an ideal

(11) \( B = \{n, J, J\} \)

which measures the failure of \( n \) to lie in \( N_l(J). \) Here \( B \) is spanned by all associators
\( [n, x, y] \) for \( x, y \in J, \) where

(12) \( [n, x, y] = [n, y, x] \)

since \( 0 = [n, x, y] + [x, y, n] + [y, n, x] = [n, x, y] - [n, y, x] + 0 \) by (4), (3), and
the definition of \( n \in N_m(J). \) Thus \( [n, x, y] = \frac{1}{2}([n, x, y] + [n, y, x]) \) results from
linearizing \([n, x, x], \) so

(13) \( \{n, x, x\} \) spans \( B. \)

To see \( B \) is actually an ideal, note \( JB \) is spanned by all \( y[n, x, x] = -x[n, y, x] + [n, xy, x] \) (by (1), (9)) = \(-x[n, y, x] + [n, xy, x] \) (by (12)) = \(-\frac{1}{2}[n, x^2, y] + [n, xy, x] \) (by (1)) which is an ideal.

Now by (13), \([n, B, J] \) is spanned by elements \([n, [n, x, x], y] y = -[n, x, y], n \) y \([y, n, [n, x, x]] \) (by (4)) = \( D_nD_{n,x}(y) + 0 \) (by (9)) = \( \{D_{n,x}, D_{n,x} - D_{n,x}(n), x\} \)

(by (10)) = 0 (since \( D_{n,x}(n) = 0 \)), so

(14) \( [n, B, J] = 0. \)

Then \( 0 = [n, B, J] \supset [n, U_B J, J] = U_{[n, B, J], B} J + U_B [n, J, J] \) (by (9), (10)) = \( 0 + U_B B \) implies \( U_B B = 0. \) If \( J \) is semiprime then \( B = \{n, J, J\} = 0 \) by (11) and
\( n \in N_l(J). \)

If \( J \) is merely \( D\)-semiprime we must work a little harder. We claim \( C = [N_m(J), J, J] = \sum [n, J, J] \) is a trivial \( D\)-invariant ideal. \( C \) is an ideal since each
\( [n, J, J] \) is by (11), and it is \( D\)-invariant by (2) and the fact that \( N_m(J) \) is \( D\)-invariant.

As in (14) we have

(14') \( [n, C, J] = 0 \)

for any \( n \in N_m(J) \) since by (12) this is spanned by all \([n, [n', x, x, x]], y] = [[n, n', y], x, x]
\([x, x] + [n', [n, x, y], x] + [n', x, [n, y, x]] \) (by (9), (2)) = \( 0 + 2[n', [n, y, x], x] \) (by (12) and
\( n' \in N_m(J) \) = \( 2([n', [n, x, y]], x] - 2n' \{[n, x, y], x] \) (by (12)) = \( [n, x^2, y] - [n, n', y, x^2] \)
(by (1), (9)) = 0 (by (12)). Then \( 0 = [n, C, J] \supset [n, C J, J] \) = \([n, C, J] \supset J + \([n, J, J] \) (by (1)) = \( 0 + C[n, J, J] \) (by (14')); this holds for all \( n \in N_m(J), \) so
\( 0 = CC \) and \( C \) is trivial as well as \( D\)-invariant. If \( J \) is \( D\)-semiprime this forces
\( C = 0, \) so \([N_m(J), J, J] = 0 \) and \( N_m(J) \subset N_l(J). \) □
It is easy to give examples to show that the semiprimeness hypothesis is needed. If we take \( J = F/K \) for \( F \) the free Jordan algebra on \( x, y \), \( K \) the ideal generated by all \([a, y, b] \) for \( a, b \in F \), then \( n = y + K \) is middle-nuclear in \( J \) by construction but is not central: \([n, J, J] \neq 0 \) since \([y, x, x] \notin K \) (\( K \) is graded, vanishing in total degrees < 3, and in degree \( x^2y^1 \) it is spanned by \([x, y, x] = 0 \), so \([y, x, x] \notin K \)).

We can also easily give special examples \( J = A^+ \) for \( A \) an associative algebra. Here \( n \in N_m(J) \Leftrightarrow [[n, J], J] = 0 \) and \( n \in N_l(J) \Leftrightarrow [[n, J], J] = 0 \). If \( A = \Phi E_{11} + \Phi E_{12} + \Phi E_{22} \) consists of all upper triangular \( 2 \times 2 \) matrices over \( \Phi \), then 
\[
[J, J] = \Phi E_{12}, \quad \text{so } n = E_{12} \text{ lies in } N_m(J), \text{ yet } n \notin N_l(J) \text{ since } [[n, E_{11}], E_{11}] = -[E_{12}, E_{11}] = E_{12} \neq 0.
\]
Note that \( n \) falls in the trivial ideal \( B = \Phi E_{12} \).

Another example is the algebra \( J(Q, c) \) determined by a quadratic form \( Q \); here \( n \in N_m(J) \Leftrightarrow [[n, J], J] = 0 \) and \( n \in N_l(J) \Leftrightarrow [[n, J], J] = 0 \). If \( A = E_{-1} + E_{22} + \Phi \) consists of all upper triangular \( 2 \times 2 \) matrices over \( \Phi \), then 
\[
[J, J] = E_{12}, \quad \text{so } n = E_{12} \text{ lies in } N_m(J), \text{ yet } n \notin N_l(J) \text{ since } [[n, E_{11}], E_{11}] = -[E_{12}, E_{11}] = E_{12} \neq 0.
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[J, J] = \Phi E_{12}, \quad \text{so } n = E_{12} \text{ lies in } N_m(J), \text{ yet } n \notin N_l(J) \text{ since } [[n, E_{11}], E_{11}] = -[E_{12}, E_{11}] = E_{12} \neq 0.
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\]
We can also give special examples \( J = A^+ \) for \( A \) an associative algebra.

Without reference to middle nuclei we can establish

**Theorem.** If \( J \) is a semiprime linear Jordan algebra then \( C(I) \subset C(J) \) for any ideal \( I \triangleleft J \). More generally, \( C(J) \subset C(J) \) as soon as \( J \triangleleft J \) and \( J \) contains no trivial ideals of \( J \) invariant under all derivations of \( J \) which map \( J \) into itself.

**Proof.** The first assertion follows from the second, since by a result of Slōnko [4] if \( J \) is semiprime so is any \( I \triangleleft J \). Assume \( J \) is \( D \)-semiprime in \( J \) in the above sense; for convenience we may assume that \( J \) is unital. We must show that if \( c \in C(J) \) then \( c \in C(J) = N_l(J) \), i.e. \([c, J, J] = 0 \).

All the derived ideals \( J^{(n)} \) (where \( J^{(0)} = J, J^{(n+1)} = U_{J^{(n)}}, J^{(n+2)} \)) are invariant under the indicated derivations and remain ideals in \( J \), as are their annihilators \( J^{(n)} \) (if \( B \triangleleft J \) is invariant so is \( B \) since \( B \triangleleft J \) is a subalgebra of \( J \)). Moreover, \( B = J \cap J^{(n)} \) are solvable: \( B^{(n)} \subset J^{(n)} \cap J^{(n)} \). If \( J \) contains no trivial invariant ideals then it contains no solvable invariant ideals of \( J \), so \( B = 0 \): 

\[
\text{(15)} \quad \text{if } J \text{ is } D\text{-semiprime in } \tilde{J} \text{ then } J \cap J^{(n)} = 0.
\]

If \( D(J^{(n)}) = 0 \) then \( D(J) \subset J^{(n+1)} \subset J^{(n+1)} \subset J^{(n)} \subset \{D(J), J^{(n)}\} \) (because \( \{d, x, y, z, \tilde{a}, \tilde{b}\} = \{d, x, \tilde{a}, \tilde{y}, y, z\} \) where \( x, y \in J^{(n)} \triangleleft \tilde{J} \).

In particular, for \( D = D_{c, J} \) as in (9) we see \([c, J, J] = 0 \) (by \( c \in N_l(J) \)) implies \([c, J, J] = 0 \), hence for \( D = D_{c, J} \) we see \([c, J^2, J] = -[J^2, J, c] - [J, J, J^2] \) (by (4)) \( \subset [c, J, J^2] + [J, c, J] \) (by (3) and linearized (6)) \( \subset [c, J, J] + [J, c, J] \) (by \( J \triangleleft J \) = 0 (by the above and \( c \in N_m(J) \)) implies \([c, J, J] = 0 \) as desired. □
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