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ON LITTLEWOOD'S CONJECTURE FOR UNIVALENT FUNCTIONS[1]

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ABSTRACT. The uniform asymptotic theory of functionals on S is investigated. We prove that Littlewood's conjecture is equivalent to the asymptotic Bieberbach conjecture of Hayman.

1. Introduction. Let S denote the class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

regular and univalent in the unit disk $\{|z| < 1\}$. Now let $A_n = \sup\{|a_n| : f \in S\}$. Hayman [8] showed that $A_n/n \to K_0$ and conjectured that $K_0 = 1$. Nehari [11] proved that for $f(z) \in S$

$$|a_n| \leq 4nK_0 \operatorname{dist}(0, \mathbf{C} - f(|z| < 1)), \tag{1}$$

i.e. Hayman's conjecture implies Littlewood's conjecture [10] (see also [1]). In this paper we use some recent results to show the converse, as well as a number of related results.

2. Background results. Hayman [9] considered the class[2] H of functions $\phi(z)$ univalent and nonzero on $\{\operatorname{Re}(z) > 0\}$ which satisfy

$$\lim_{x\to\infty} x^2|\phi(x)| = \mu \leq 1. \tag{2}$$

The class H arises from limits of sequences $f_n(z) = \sum_{m=1}^{\infty} a_{n,m} z^m \in S$ such that $\underline{\lim}\, n^{-2}|f_n(1-1/n)| > 0$. For such sequences we have

$$n^{-2} f_n(1 - z/n) \to \phi(z) \in H \tag{3}$$

locally uniformly, at least on a subsequence. Conversely for every $\phi(z) \in H$ there is a sequence $f_n \in S$ with property (3). The other fundamental result is that

$$\frac{a_{n,m}}{n} \to T_a(\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(x+iy) e^{a(x+iy)}\, dy, \quad \text{as } n \to \infty, m/n \to a > 0.$$

Also, Hayman shows that

$$\sup_{\phi\in H} |T_a(\phi)| = K_0 a. \tag{4}$$

We also need some other facts about the asymptotic theory of univalent functions. Define

$$\rho(r) = \frac{(1-r)^2}{r}|f(r)|, \qquad f \in S,$$

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[2]This terminology is due to Bombieri [1].

then $\rho(r)$ decreases to a number $\alpha \leq 1$. Hayman's [8] famous asymptotic result is that if $\alpha > 0$ then

$$|a_n|/n \to \alpha. \tag{5}$$

In [9] he states the following generalization. For every positive $\epsilon > 0$ there is a positive δ such that whenever

$$\rho(1 - x_1/n) \geq (1 - \delta)\rho(1 - x_2/n) \tag{6}$$

we have $||a_m|/m - \rho(1 - 1/m)| \leq \epsilon$, $n/x_2 \leq m \leq n/x_1$. Here $x_1 \leq x_2$ are regarded as being fixed positive numbers. (This is the quantitative version of the Hayman asymptotic theorem given in Brown and Hamilton [3].)

Recently Bombieri [2] developed FitzGerald type inequalities for H and immediately deduced (5). We conclude this paper with a short proof of (5) making use of elementary ideas from H theory.

3. Main results. First we prove

THEOREM 1. *Suppose that $\phi(z) \in H$ satisfies $\lim_{x\to\infty} x^2|\phi(x)| = 1$, and $f_n \in S$ is a sequence having property* (3). *Then for every $\epsilon > 0$, there exists γ, N such that*

$$|f_n(r)| \geq (1 - \epsilon)r/(1 - r)^2, \tag{7}$$

for $r \leq 1 - \gamma/n$, $N \leq n$.

Our assumptions show that $\gamma^2|\phi(\gamma)| \geq 1 - \epsilon/4$ for some γ and also by (3)

$$\left|n^{-2} f_n\left(1 - \frac{\gamma}{n}\right) - \phi(\gamma)\right| \leq \frac{\epsilon}{4\gamma^2} \quad \text{for } N \leq n.$$

Consequently if $R = 1 - \gamma/n$, $n \geq N$, $((1 - R)^2/R)|f_n(R)| \geq (1 - \epsilon)$, which proves (7) as $\rho(t)$ is nonincreasing.

REMARK 1. Hamilton [6] uses (7) to prove that if $|a_2| \leq 1.8\ldots$ then $|a_n| \leq n$, $n \geq N$, and in fact problems of this type are essentially equivalent to bounding K_0.

COROLLARY 1. *With the above hypothesis $|a_{n,2}| \to 2$.*

From (6) we have that

$$(1 - \epsilon)\{r + 2r^2 + \ldots\} \leq r + |a_{n,2}|r^2 + \ldots.$$

Thus for any $\epsilon > 0$, small $r - \epsilon + 2(1 - \epsilon)r^2 \leq |a_{n,2}|r + o(r)$. This implies that $\underline{\lim}\, |a_{n,2}| = 2$. The reverse inequality is the well-known bound $|a_{n,2}| \leq 2$. In fact by (7), $a_{n,2} \to 2$.

REMARK 2. FitzGerald [4] proved a similar result for the case that f_n was a sequence in S such that $a_{n,n}/n \to K_0$. He used a construction of Landau, peaking properties of series, Hayman's results on H and the maximal property of $|a_{n,n}|$.

REMARK 3. The corollary shows that $f_n(z) \to z/(1 - z)^2$ locally uniformly. In fact the theorem shows that the convergence is much faster than that on the positive axis. Later we shall give precise estimates of convergence for the interesting case $a_{n,n} \sim A_n$.

FitzGerald's result, considered with (4), shows that if ϕ maximizes $|T_1|$ and $n^{-2}f_n(1 - z/n) \to \phi(z)$ then $a_{n,2} \to 2$. Now let Λ be a continuous linear functional defined on the space A of analytic functions on $\{\mathrm{Re}(z) > 0\}$, with the usual topology induced by convergence on compact subsets of $\{\mathrm{Re}(z) > 0\}$. $H \cup \{0\}$ is a compact subset of A. Also we assume that Λ is nonconstant, i.e. nonzero on H. A support point $\phi_0 \in H$ is defined by $\Lambda(\phi_0) = \max\{|\Lambda(\phi)| \colon \phi \in H\}$ for some nonconstant Λ.

LEMMA 1. *Suppose that ϕ_0 is a support point then*

$$\lim_{x\to\infty} x^2|\phi(x)| = 1. \tag{8}$$

PROOF. For if $\mu = \lim_{x\to\infty} x^2|\phi_0(x)| < 1$, then $\phi_0/\mu \in H$ but $|\Lambda(\phi_0/\mu)| > |\Lambda(\phi_0)|$ which is impossible. Thus we have proved

THEOREM 2. *Suppose that for $f_n \in S$, $n^{-2}f_n(1-z/n)$ converges to a support point ϕ_0 of H. Then $a_{n,2} \to 2$.*

From this we deduce the proof that Littlewood's conjecture implies Hayman's.

THEOREM 3. *Let K be a constant such that*

$$|a_n| \le 4nK\,\mathrm{dist}(0, \mathbf{C} - f\,(|z|<1)) \tag{9}$$

for any $f \in S$. Then $K_0 \le K$.

PROOF. Suppose that $\phi \in H$ satisfies $|T_1(\phi)| = K_0$, and $n^{-2}f_n(1-z/n) \to \phi(z)$, $f_n \in S$. Then $a_{n,n}/n \to K_0$, while by Theorem 2 $a_{n,2} \to 2$. The latter implies that $\mathrm{dist}(0, \mathbf{C} - f_n(|z|<1)) \to \frac{1}{4}$, since f_n converges to $z/(1-z)^2$ locally uniformly and hence by the Carathéodory kernel theorem. Thus by (9), as $|a_{n,n}|/n \to K_0$ while $4K\,\mathrm{dist}(0, \mathbf{C} - f_n\,(|z|<1)) \to K$.

REMARK 4. The equivalence of the two conjectures follows immediately from (1).

2. Characterization of functionals on H. It is of interest to determine the connection between linear functionals λ_n on S and functionals Λ on H. Obviously every sequence λ_n does not lead to the consideration of a functional Λ on H. For instance, if $\lambda_n(f) = f(\frac{1}{2})$, or $\lambda_n(f) = f(1-1/n)$. In the first case we see that some sort of growth condition is needed, while in the second case we require a normalization so that $\lambda_n(f_n)$ remains bounded. Basically the major contribution to $\lambda_n(f)$ must come from the neighborhood of $1-1/n$.

Now any bounded linear functional Λ on H may be regarded as a functional on $\{T_a(\phi)\colon \phi \in H\}$. Relation (4) shows that this set is naturally embedded in the space of all continuous functions $s(a)$ on $(0,\infty)$ such that $\|s\| = \max_a |s(a)/a| < \infty$. Then the Riesz representation theorem shows that

$$\Lambda(\phi) = \int_0^\infty \left(\frac{T_a}{a}\right) d\mu(0) \tag{10}$$

for some finite complex measure μ. Next we prove

THEOREM 4. *For any functional Λ on H there is a sequence λ_n on S such if f_n is a sequence in S and $n^{-2}f_n(1-z/n) \to \phi(z)$ then $\lambda_n(f_n) \to \Lambda(\phi)$.*

PROOF. We define $b_m^{(n)} = \mu((m-1)/n, m/n)$ and

$$\lambda_n\left(\sum_{m=0}^\infty a_m z^m\right) = \sum_{m=0}^\infty \frac{a_m}{m} b_m(n) \to \int_0^\infty \frac{T_a}{a}\,d\mu \quad \text{as } n\to\infty. \tag{11}$$

The converse is also true, in the following sense. Suppose that λ_n is a sequence of functionals such that $\overline{\lim}\{\max_s |\lambda_n(f)|\} < \infty$. Then there is a subsequence of n such that "λ_n converges to some functional Λ on H". By this we mean if

$n^{-2}f_n(1-z/n) \to \phi(z)$ then $\lambda_n(f_n) \to \Lambda(\phi)$. This is just a consequence of the weak compactness of the unit ball in the space of functionals, and the further condition

$$(12) \qquad f_n \in S, \qquad n^{-2}f_n(1-z/n) \to 0 \quad \text{then } \lambda_n(f_n) \to 0.$$

Under these conditions then $\overline{\lim}\max_S|\lambda_n(f)| = \max_H|\Lambda(\phi)|$.

REMARK 5. Similar results hold for the real part of λ_n. For example if $f_n \in S$ maximizes $\operatorname{Re} e^{i\theta}f(1-1/n)n^{-2}$ then $a_{n,2} \to 2$.

REMARK 6. In fact our results hold for homogenous functionals. Actually not every linear functional on H is represented by (10). For instance in [**7**] it is shown that T_a is continuously differentiable, but T'_a is unbounded. Thus not every functional of interest is bounded.

5. Further properties of H. It is a consequence of an inequality of FitzGerald (see [**4**]) that for every positive ϵ there is positive δ, η such that $|a_n| \geq (1-\delta)K_0 n = |a_m| \leq (1+\epsilon)m$, $m \leq \eta n$. We now prove

THEOREM 5. *Let Λ be a nonconstant functional on H and define $K = \max_H|\Lambda(\phi)|$. Then for every positive ϵ there is positive δ, η such that*

$$\Lambda(\phi) > (1-\delta)k = |T_a(\phi) - a| \leq \epsilon a, \qquad a \leq \eta.$$

REMARK 7. Even the case $\Lambda = T_1(\phi)$ is stronger than previous results.

PROOF. Suppose that $G = \{\phi \in H\colon |\Lambda(\phi)| \geq K(1-\delta)\}$, and $\tau(x) = \min\{x^2|\phi(x)|\colon \phi \in G\}$. Now, (see [**9**, p. 5]) $x^2|\phi(x)|$ increases. Thus $\tau(x)$ is nondecreasing.

Next we show that $\lim_{x\to\infty}\tau(x) \geq 1-\delta$. First suppose that $\lim_{x\to\infty}\tau(x) = \mu < (1-\delta)$. Then there is a sequence $\phi_n \in G$, $x_n \to +\infty$ such that $x_n^2|\phi_n(x_n)| \to \mu$. Now some subsequence of ϕ_n converges to a function $\phi \in G$. As $x^2|\phi(x)|$ and $x^2|\phi_n(x)|$ are nondecreasing $\lim x^2|\phi(x)| \leq \mu$. On the other hand $|T_1(\phi)| \geq K(1-\delta)$. However $\phi/\mu \in H$ and hence $|T_1(\phi/\mu)| \geq K(1-\delta)/\mu > K$ which is impossible. Thus there is a positive y such that for $x \geq y$,

$$(13) \qquad x^2|\phi(x)| \geq 1-2\delta, \qquad \phi \in G.$$

We now obtain the analogy of (6) for H. Let $\sigma(x) = x^2|\phi(x)|$. Let $f_n \in S$ satisfy $n^{-2}f_n \to \phi$. Then $\rho(1-x/n, f_n) \to \sigma(x)$. Thus (6) becomes $\sigma(x_1) \geq (1-\delta)\sigma(x_2) = |\,|Ta/a| - \sigma(1/a)| \leq \epsilon, 1/x_2 \leq a \leq 1/x_1$. then (13) shows these conditions are satisfied as $x_2 \to \infty$. Thus as $|\sigma(1/a) - 1| \leq 2\delta$ we obtain our result for $a \leq 1/y$.

REMARK 8. This result may be interpreted to show that for any positive ϵ there exists δ, η such that $|a_m|/m - 1 \leq \epsilon$ whenever $|a_n|/n = K_0(1-\delta)$ and $m \leq \eta n$ or $m \geq \eta^{-1}n$. Thus the Bieberbach conjecture fails on a relatively short interval. Hayman [**9**] shows that $|Ta/a| \leq 1$ except on a set of finite logarithmic measure.

6. Another proof of (5). We only need

LEMMA 2. $x^2|\phi(x)| = 1$ *if and only if* $\phi(z) = e^{i\theta}/z^2$.

This result is obtained by considering $-\frac{1}{4}\{\phi((1-z)/(1+z)) - 1\} \in S$ via the corresponding result for S.

Without loss of generality $\rho(r) \to \alpha$, $0 < \alpha < 1$, as the cases $\alpha = 0$, 1 are trivial. Now let $f_n = f$, then $\rho(r) \to \alpha$ implies that, as $n^{-2}|f_n(1-1/n)| > \alpha$, $n^{-2}f_n(1-z/n) \to \phi(z)$ on some subsequence, then $|\phi(x)| = \alpha/x^2$. Thus the lemma shows that $\phi(z) = e^{i\theta}\alpha/z^2$ for some $\phi \in H$ (by considering ϕ/α). Thus $a_n/n \to T_1(e^{i\theta}\alpha/z^2) = e^{i\theta}\alpha$ along the same subsequence such that $n^{-2}f_n \to \phi(z)$. Thus for any subsequence such that $n^{-2}f_n$ converges we have $|a_n|/n \to \alpha$. Thus $|a_n/n| \to \alpha$ as $n \to \infty$.

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