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ON THE LOCATION OF THE ZEROS OF THE DERIVATIVE OF A POLYNOMIAL

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ABSTRACT. We show that nonreal zeros of a real polynomial $P(z)$ sometimes induce nonreal zeros of $P'(z)$.

1. Introduction. We prove two results in the geometry of the zeros of polynomials [**2**]. In particular, we study how the distribution of the zeros of a real polynomial may force (or prevent) the existence of a nonreal zero of its derivative. Our investigation was originally motivated by certain considerations about the zeros of Riemann's xi-function (and of its derivative) which arise in analytic number theory (see [**1**]). We note that our method applies also to real entire functions of order ≤ 1.

2. Notation. For simplicity we will work with polynomials though the proofs are valid in a more general situation. We assume henceforth that P is a polynomial with real coefficients. Generic real zeros of P will be denoted r and complex conjugate pairs are w and $\bar{w}$ where $w = u + iv$. (To indicate specific zeros we use subscripts.) We associate to w the Jensen disc

$$J_w = \{z\colon |z - u| \leq v\}.$$

A well-known theorem of Jensen [**2**] asserts that if $P'(z) = 0$ then either z is real or $z \in J_w$ for some w. This follows immediately from the fact that if $f(z) = (z - w)(z - \bar{w}) = (z - u)^2 + v^2$, then

$$\operatorname{Im} \frac{f'}{f}(z) = -2y \frac{|z - u|^2 - v^2}{|(z - u)^2 + v^2|^2}$$

(so that $\operatorname{Im} P'(z)/P(z) \neq 0$ if z is nonreal and outside every Jensen disc).

We also write $z = x + iy$.

3. Statement of results. Our first theorem asserts that if a real polynomial P has real zeros "near" to the projection of a complex zero w_0 of P onto the real axis, then P' has a nonreal zero, provided that w_0 is, in a certain sense, isolated. By "near" we mean within a distance of the order of magnitude of $|\operatorname{Im} w_0|$.

THEOREM 1. *Let $n \geq 1$ be fixed and let*

$$c_n = \begin{cases} 0.5 & \text{if } n = 1, \\ \frac{1}{2}(\sqrt{9n^2 - 8n} - n)^{1/2} & \text{if } n > 1. \end{cases}$$

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Suppose that $P(w_0) = P(r_1) = \cdots = P(r_n) = 0$ and that $|u_0 - r_i| < c_n|v_0|$ for $1 \le i \le n$. Suppose further that $J_w \cap J_{w_0} = \emptyset$ where w ranges over the nonreal zeros of P with $w \ne w_0, \bar{w}_0$. Then P' has a nonreal zero inside J_{w_0}.

We can obtain a more precise result if the zeros of P are symmetric about the y-axis.

THEOREM 2. *Suppose that P is an even function, $P(0) \ne 0$, and $P(iv_0) = 0$ where $v_0 > 0$. Let r_0 be the smallest real zero of P such that $r_0 \ge v_0$. Suppose that $J_w \cap [-r_0, r_0] = \emptyset$ where w ranges over the nonreal zeros of P with $w \ne \pm iv_0$. Define*

$$K = \sum_r \frac{1}{r^2} + \sum_w \frac{u^2 - v^2}{(u^2+v^2)^2}.$$

Then P' has a nonreal zero inside J_{iv_0} if and only if $K > 0$.

It follows from this theorem that c_n of Theorem 1 is asymptotically best possible. For if the zeros of P consist of n (where n is even) real zeros all at $\pm r_0$ and a pair of complex zeros at $\pm iv_0$ then Theorem 2 asserts that P' has only real zeros if

$$K = \frac{n}{r_0^2} - \frac{2}{v_0^2} < 0,$$

that is, if $r_0 > \sqrt{n/2}\,v_0$; Theorem 1 asserts that P' has a nonreal zero if $r_0 < c_n v_0$. Since $c_n \sim \sqrt{n/2}$ as $n \to \infty$, Theorem 1 is nearly sharp.

4. Proofs. We now prove Theorem 1. We may assume without loss of generality that $u_0 = 0$ so that $w_0 = iv_0$, $v_0 > 0$. Then

$$P(z) = (z^2 + v_0^2)(z - r_1)\cdots(z - r_n)Q(z)$$

where Q is a polynomial with real coefficients. Consider the contour

$$\begin{aligned} C_\epsilon &= \{z\colon |z - i\epsilon| = v_0 \text{ and } y > \epsilon\} \cup \{z = x + i\epsilon\colon\ -v_0 \le x \le v_0\} \\ &= C_\epsilon^U + C_\epsilon^L \end{aligned}$$

where ϵ is a small positive number. We will show that if ϵ is small enough then $\operatorname{Im} P'(z)/P(z) < 0$ on C_ϵ. Then the change in the argument of $P'(z)/P(z)$ as z varies around C_ϵ is 0. Hence by the argument principle the number of zeros of P' inside C_ϵ is equal to the number of zeros of P inside C_ϵ, namely one ($P(iv_0) = 0$), which shows the result.

By an easy calculation, $\operatorname{Im} P'(z)/P(z) = -yF(z) + \operatorname{Im} Q'(z)/Q(z)$ where

$$F(z) = 2\frac{|z|^2 - v_0^2}{|z^2 + v_0^2|^2} + \sum_{i=1}^n \frac{1}{|z - r_i|^2}.$$

By the hypothesis about the Jensen discs of P, $\operatorname{Im} Q'(z)/Q(z) < 0$ for z on C_ϵ, if ϵ is small enough. Also, $F(z) > 0$ for z on C_ϵ^U for any $\epsilon > 0$. Finally we will show that $F(z) > 0$ for z on C_0^L which implies that $F(z) > 0$ for z on C_ϵ^L, if $\epsilon > 0$ is sufficiently small. Hence we must show that

$$F(x) = F(x, r_1, r_2, \ldots, r_n) = 2\frac{x^2 - v_0^2}{(x^2 + v_0^2)^2} + \sum_{i=1}^n \frac{1}{(x - r_i)^2} > 0$$

for $|x| \leq v_0$ and $|r_i| \leq c_n v_0$. We consider the minimum M of F on this set. By symmetry, a minimum value of F will occur in $x \geq 0$. Since $\partial F/\partial r_i$ is never 0, the minimum for F occurs when each $r_i = \pm c_n v_0$. Therefore,

$$M = \min_{0 \leq x \leq v_0} 2\frac{x^2 - v_0^2}{(x^2 + v_0^2)^2} + \frac{n}{(x + c_n v_0)^2}.$$

Let $f(x) = 2(x^2 - v_0^2)(x + c_n v_0)^2 + n(x^2 + v_0^2)^2$. Then to prove $M > 0$ it is sufficient to show that $f(x) > 0$ for $0 \leq x \leq v_0$ or, equivalently, that

$$g(x) = 2(x^2 - 1)(x + c_n)^2 + n(x^2 + 1)^2 > 0$$

for $0 \leq x \leq 1$. For $n = 1$ and $c_1 = 0.5$ this is easily shown by Sturm's Theorem (see [**3**]). For $n > 1$ we observe that

$$\begin{aligned} g(x) &= (n + 2)x^4 + 4c_n x^3 + 2(n - 1 + c_n^2)x^2 - 4c_n x + (n - 2c_n^2) \\ &\geq 2(n - 1 + c_n^2)x^2 - 4c_n x + (n - 2c_n^2) \end{aligned}$$

for $x \geq 0$ if $c_n \geq 0$. The last expression in positive for all x if

$$0 > 16c_n^2 - 8(n - 1 + c_n^2)(n - 2c_n^2) = 8(2c_n^4 + nc_n^2 - n(n - 1))$$

which holds if $c_n^2 \leq \left(-n + \sqrt{n^2 + 8n(n-1)}\right)/4$.

To prove Theorem 2 we introduce an auxiliary polynomial

$$P_t(z) = (z^2 + t^2 v_0^2) \prod_r (z - r) \prod_{\substack{w \neq iv_0 \\ v > 0}} ((z - u)^2 + v^2)$$

so that P_t and P have the same zeros except that P is zero at $\pm iv_0$ while P_t is zero at $\pm itv_0$. Let C_ϵ be the circle of radius $r_0 + \epsilon$ with center at the origin. Let $\epsilon > 0$ be so small that $[-r_0 - \epsilon, r_0 + \epsilon] \cap J_w = \emptyset$ for all $w \neq \pm iv_0$. Then (as in Jensen's Theorem) $\operatorname{Im} P_t'(z)/P_t(z)$ is zero only at the two real points of C_ϵ, for $0 \leq t \leq 1$. Therefore the change in $\arg P_t'(z)/P_t(z)$ on C_ϵ is 0 or $\pm 2\pi$ so that, by the argument principle, the number of zeros of $P_t'(z)$ inside C_ϵ differs from the number of zeros of $P_t(z)$ inside C_ϵ by at most one, for $0 \leq t \leq 1$. By Rolle's Theorem, $P_t'(x)$ has an odd number of real zeros strictly between consecutive real zeros of $P_t(x)$. To help envision the proof, imagine that we are watching a movie of the zeros of P_t' as t moves from 0 to 1. When $t = 0$, $P_t(z) = P_0(z)$ has only real zeros inside or on C_0 (the circle of radius r_0 with center the origin). Moreover, by Jensen's Theorem, $P_0'(z)$ has only real zeros inside C_0, and by the above remarks $P_0'(x)$ has precisely one real zero (counting multiplicities) between consecutive real zeros of $P_0(x)$, for $-r_0 \leq x \leq r_0$. For if $P_0'(x)$ had three (or more) zeros between consecutive real zeros r and r' of $P_0(x)$ then (since P_0' is an odd function) P_0' would have three or more zeros between $-r'$ and $-r$ and the total number of zeros of P_0' in $[-r_0, r_0]$ would exceed that of P_0 by at least three, in contradiction to what we deduced from the argument principle. Thus, when the movie starts the zeros of P_t' inside C_0 are real and simple (except if P_0 has a zero of multiplicity $m \geq 3$ at some $r \neq 0$ which causes a zero of P_0' of multiplicity $m - 1$ at r) and interlace the zeros of P_0 (all of which are real). Also P_0 has a double zero at the origin.

Now the zeros of P_t' are continuous functions of t. Therefore no new nonreal zeros of P_t' can enter C_0 as t varies from 0 to 1, for nonreal zeros of P_t' must remain in the various Jensen discs, none of which intersects C_0 (except, of course, $J_{\pm iv_0}$).

Also, a real zero of P'_t cannot coalesce with a (previously distinct) real zero of P. For if $r \neq 0$ is a zero of P of multiplicity $m > 0$ then r is a zero of P_t of multiplicity m and therefore a zero of P'_t of multiplicity (precisely) $m - 1$. Thus, as the movie progresses ($0 \leq t \leq 1$) no new zeros of P'_t enter or leave C_0 and the real zeros of P'_t which are between the consecutive zeros r, r' of P with $r < r' < 0$ (or $r > r' > 0$) must remain real and between r and r'.

Let $0 < r_1 \leq r_0$ with $P(r_1) = 0$ and $P(x) \neq 0$ for $-r_1 < x < r_1$. The zeros of P'_t in $(-r_1, r_1)$ will behave differently from those described above. When $t = 0$, there are three zeros of P'_t in $(-r_1, r_1)$: one in $(-r_1, 0)$, one in $(0, r_1)$ and one at the origin. The one at the origin remains there for all t. The other two will vary with t: call them $S_t^{\pm}$, with $S_0^{+} > 0$. It is clear from what has been so far described that the only way for $P'(z)$ to have nonreal zeros inside C_0 is if S_t^{+} and S_t^{-} coalesce at the origin and become (and remain) purely imaginary conjugates. We will now show that if $K \leq 0$ then $S_1^{\pm}$ are real while if $K > 0$ then $S_1^{\pm}$ are purely imaginary nonreals.

If $S_1^{\pm}$ are nonreal then for some t_0, $0 < t_0 < 1$, P'_{t_0} has a triple zero at the origin. Hence if $P''_t(0) \neq 0$ for $0 < t < 1$ then $S_1^{\pm}$ are real. But an easy calculation gives

$$P''_t(x) = P'_t(x) \cdot \frac{P'_t}{P_t}(x) + P_t(x)\frac{d}{dx}\left(\frac{P'_t}{P_t}(x)\right)$$

so that $P''_t(0) = -P_t(0)K(t)$ (since $P'_t(0) = 0$) where

$$K(t) = \frac{2}{t^2 v_0^2} + \sum_r \frac{1}{r^2} + \sum_{w \neq \pm i v_0} \frac{u^2 - v^2}{(u^2 + v^2)^2}.$$

Since $P_t(0) \neq 0$ it follows that $K(t) \neq 0$ for $0 < t < 1$ implies that $S_1^{\pm}$ are real. Clearly K is an increasing function of t which is negative for all sufficiently small $t > 0$. Therefore, if $K = K(1) < 0$ then $K(t) < 0$ for all t, $0 < t < 1$. Hence $S_1^{\pm}$ are real in the event that $K < 0$, as was asserted in the theorem.

To prove the other assertion we consider

$$P_\infty(z) = \prod_r (z - r) \prod_{\substack{w \neq \pm i v_0 \\ v > 0}} ((z - u)^2 + v^2),$$

so that $P_\infty(z) = P(z)/(z^2 + v^2)$. With the same argument we used above we conclude that within C_ϵ the zeros of P'_∞ are all real and interlaced with the zeros of P_∞. In particular, if $-r_1 < x < r_1$ then $P'_\infty(x) = 0$ only when $x = 0$. Now if t is sufficiently large (we no longer require $t \leq 1$), then P'_t and P'_∞ will have the same number of zeros in $-r_1 < x < r_1$. This is because

$$\left|\frac{P'_\infty}{P_\infty}(z) - \frac{P'_t}{P_t}(z)\right| = \left|\frac{2z}{z^2 + t^2 v_0^2}\right|$$

so that $P'_t(x)/P_t(x)$ is bounded away from 0 when $P'_\infty(x)/P_\infty(x)$ is (for large t) while in a small neighborhood of the origin (which is the only zero of $P'_\infty(x)/P_\infty(x)$ in $-r_1 < x < r_1$) P'_t/P_t and P'_∞/P_∞ have the same number of zeros by Rouché's Theorem. Therefore, for all sufficiently large t, the zeros $S_t^{\pm}$ are not real. Nonreal zeros of P'_t which are not purely imaginary occur in sets of four, $(\pm w, \pm \bar{w})$, since P'_t is an odd, real function. Therefore, as soon as t is large enough so that $S_t^{\pm}$ are no longer real they must be purely imaginary and the transition occurs at some t_0

with $-P_{t_0}(0)K(t_0) = P''_{t_0}(0) = 0$. But if $K(1) = K > 0$ then $K(t) > 0$ for all $t \geq 1$, since $K(t)$ is an increasing function. Therefore, if $K(1) > 0$ then there is a $t_0 < 1$ such that $K(t_0) = 0$ and $S_t^{\pm}$ are nonreal for all $t > t_0$. In particular $S_1^{\pm}$ are nonreal and are inside J_{iv_0}, by Jensen's Theorem. This proves Theorem 2.

5. An open question. We wonder to what extent the hypothesis in Theorem 1, $J_w \cap J_{w_0} = \emptyset$, is needed. For the application we have in mind we would like to be able to remove this hypothesis in the case that all the zeros of $P(z)$ are in a strip

$$-c_0 < \mathrm{Im}\{z\} < c_0$$

and $|v_0| > \frac{1}{2}c_0$. We pose the question whether or not a similar conclusion to Theorem 1 can be drawn in this case.

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