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# AN INFINITE CLASS OF PERIODIC SOLUTIONS OF PERIODICALLY PERTURBED DUFFING EQUATIONS AT RESONANCE

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ABSTRACT. In this paper, by using a generalized form of the Poincaré-Birkhoff Theorem, we demonstrate that the Duffing equation

$$\frac{d^2x}{dt^2} + g(x) = p(t) \qquad (\equiv p(t+2\pi))$$

may also admit an infinite number of $2\pi$-periodic solutions even in a resonance case.

**1.** In this paper we study the existence problem of periodic solutions for the Duffing equation

$$\frac{d^2x}{dt^2} + g(x) = p(t) \tag{1.1}$$

where $g(x)$, $p(t) \in C(R, R)$ and $p(t)$ is $2\pi$-periodic.

In the super-linear case, that is,

$$\lim_{|x|\to\infty} [x^{-1}g(x)] = \infty,$$

it has been proved in a recent paper by W. Y. Ding that (1.1) has infinitely many $2\pi$-periodic solutions [**1**].

D. E. Leach proved the existence and uniqueness of $2\pi$-periodic solution of (1.1) under the following Loud condition

$$m^2 < \lambda \le g'(x) \le \mu < (m+1)^2 \qquad (g(0) = 0), \tag{1.2}$$

with a given integer $m \ge 0$ and two such constants $\lambda$ and $\mu$ [**2**]. R. Reissig proved the existence of periodic solutions of (1.1) under a weaker condition [**3**]

$$m^2 < \lambda \le x^{-1}g(x) \le \mu < (m+1)^2, \qquad |x| \ge a > 0.$$

The crucial point of the conditions above is to exclude the resonance cases. At resonance, (1.1) may have no bounded solution [**6**]; therefore, the existence problem of periodic solutions challenges attention. A. C. Lazer and D. E. Leach, and L. Césari succeeded to prove the existence of $2\pi$-periodic solutions of (1.1) at a point of resonance provided that $h(x) = g(x) - m^2x$ is bounded and some other additional conditions are assumed [**4, 5**]. In a recent paper, the present author resolved the existence problem of $2\pi$-periodic solutions of (1.1) under a weakened version of (1.2) [**6**]:

$$m^2 \le g'(x) \le (m+1)^2 \qquad (g(0) = 0).$$

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In this paper, by using a generalized form of the Poincaré-Birkhoff Theorem due to W. Y. Ding, we will demonstrate that (1.1) may also admit an infinite number of $2\pi$-periodic solutions even in a resonance case. This shows once more the complexity of (1.1) at resonance.

**2.** We need the following hypotheses:

($H_1$) Let $g(x) \in C'(R, R)$, and let $K$ be a positive constant, such that

$$|g'(x)| \leq K, \qquad x \in R;$$

($H_2$) There exist two constants $A_0 > 0$ and $M_0 > 0$, such that

$$x^{-1}g(x) \geq A_0, \qquad |x| \geq M_0.$$

Then we consider the auxiliary equation

$$\frac{d^2z}{dt^2} + g(z) = 0 \tag{2.1}$$

and its equivalent system

$$\frac{dz}{dt} = w, \qquad \frac{dw}{dt} = -g(z). \tag{2.1'}$$

This is a planar autonomous system whose orbits are curves determined by the following equation

$$V(z, w) \equiv \tfrac{1}{2}w^2 + G(z) = c, \tag{2.2}$$

where $G(z) = \int_0^z g(u)\,du$ and $c$ is a parameter.

The hypothesis ($H_2$) obviously implies

$$\lim_{|z|\to\infty} G(z) = \infty \quad \text{and} \quad \lim_{|z|+|w|\to\infty} V(z, w) = \infty.$$

It follows that the set $V^{-1}(c)$ is compact. Furthermore, ($H_2$) yields that there exist constants $C_0 > 0$ and $A_1 > 0$ such that if $c \geq c_0$, then

$$w^2 + zg(z) \geq A_1(z^2 + w^2), \qquad (z, w) \in V^{-1}(c). \tag{2.3}$$

Note that the left-hand member of the last inequality is just the directional derivative of $V(z, w)$ along the vector $(z, w)$. Therefore, there is no critical point in $V^{-1}(c)$ for $c \geq c_0$, and $V^{-1}(c)$ is a compact one-dimensional manifold. Furthermore $V^{-1}(c)$ is star-shaped about the origin. We have thus proved

LEMMA 2.1. *If* ($H_2$) *holds, then* $V^{-1}(c)$ *is a closed curve for* $c \geq c_0$ *which is star-shaped about the origin* 0.

In the sequel, we will denote the curve $V^{-1}(c)$ by $\Gamma_c$. It follows from Lemma 2.1 that each curve $\Gamma_c$ ($c \geq c_0$) intersects the $z$-axis at two points: $(h(c), 0)$ and $(-h_1(c), 0)$, where $h(c) > 0$ and $h_1(c) > 0$ are uniquely determined by the formula

$$G(h(c)) = G(-h_1(c)) = c.$$

Let $(z(t), w(t))$ be any solution of (2.1′) whose orbit is $\Gamma_c$ ($c \geq c_0$). Clearly, this solution is periodic. Let $\tau(c)$ denote the least positive period of this solution. It follows from the equation (2.2) that

$$\tau(c) = \sqrt{2}\int_{-h_1(c)}^{h(c)} \frac{du}{\sqrt{c - G(u)}}. \tag{2.4}$$

To obtain the desired result, we need another hypothesis:

($H_3$) There exist a constant $\alpha > 0$, an integer $m > 0$, and two sequences $\{a_k\}$ and $\{b_k\}$, such that $a_k \to \infty$ and $b_k \to \infty$ as $k \to \infty$; and moreover

$$\tau(a_k) < \frac{2\pi}{m} - \alpha, \qquad \tau(b_k) > \frac{2\pi}{m} + \alpha. \tag{2.5}$$

Now we are in a position to state the main result of this paper.

THEOREM 2.1. *Assume* ($H_1$)–($H_3$) *hold. Then equation* (1.1) *has infinitely many* $2\pi$*-periodic solutions.*

A concrete example for applications of this theorem will be given in §5.

**3.** To prove Theorem 2.1, we briefly restate a generalized form of the Poincaré-Birkhoff fixed point theorem in Theorem A below [7].

Let $\mathcal{D}$ denote an annular region in the $(x, y)$-plane. The boundary of $\mathcal{D}$ consists of two simple closed curves: the inner boundary curve $\mathbf{C}_1$ and the outer boundary curve $\mathbf{C}_2$. Let $\mathcal{D}_1$ denote the simple connected open set bounded by $\mathbf{C}_1$. Consider an area-preserving mapping $T\colon R^2 \to R^2$. Suppose that $T(\mathcal{D}) \subset R^2 - \{0\}$, where 0 is the origin. Let $(\gamma, \theta)$ be the polar coordinate of $(x, y)$, that is, $x = \gamma\cos\theta$, $y = \gamma\sin\theta$. Assume the restriction $T|\mathcal{D}$ is given by

$$\gamma^* = f(\gamma, \theta), \qquad \theta^* = \theta + g(\gamma, \theta), \tag{3.1}$$

where $f$ and $g$ are continuous in $(\gamma, \theta)$, and $2\pi$-periodic in $\theta$.

THEOREM A. *Besides the above-mentioned assumptions, we assume that*

(i) $\mathbf{C}_1$ *is star-shaped about the origin;*

(ii) $0 \in T(\mathcal{D}_1)$;

(iii) $g(\gamma, \theta) > 0$ $(< 0)$, $(\gamma\cos\theta, \gamma\sin\theta) \in \mathbf{C}_1$;
$g(\gamma, \theta) < 0$ $(> 0)$, $(\gamma\cos\theta, \gamma\sin\theta) \in \mathbf{C}_2$.

*Then $T$ has at least two fixed points in $\mathcal{D}$.*

Now, let $\Gamma_{a_k}$ and $\Gamma_{b_k}$ be the curves given by Lemma 2.1, where the specified parameters $a_k$, $b_k \geq c_0$ are given by ($H_3$), for $k \geq n_0$. We may rearrange $\{a_k\}$ and $\{b_k\}$, if necessary, so that $a_k < b_k < a_{k+1}$ for $k \geq n_0$. Then, $\Gamma_{a_k}$ and $\Gamma_{b_k}$ bound an annular region $\mathcal{A}_k$, and $\Gamma_{b_k}$ and $\Gamma_{a_{k+1}}$ bound another annular region $\mathcal{B}_k$, for $k \geq n_0$.

Let $T\colon R^2 \to R^2$ be the Poincaré mapping induced by equation (1.1). It is well known that each fixed point of $T$ corresponds to a $2\pi$-periodic solution of (1.1). In the following section, we will apply Theorem A to show that $T$ has at least two fixed points in each $\mathcal{A}_k$ and $\mathcal{B}_k$ for sufficiently large $k$. As a consequence, (1.1) has an infinite class of $2\pi$-periodic solutions.

**4.** Consider the equivalent system of (1.1),

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -g(x) + p(t). \tag{4.1}$$

Let $(\bar{x}(t, x, y), \bar{y}(t, x, y))$ be the solution of (4.1) through the initial point $(\bar{x}(0), \bar{y}(0)) = (x, y)$. It is not hard to show that every such solution exists on the whole $t$-axis under condition ($H_1$). Then the Poincaré map $T\colon R^2 \to R^2$ is well defined by

$$(x, y) \mapsto (\bar{x}(2\pi, x, y), \bar{y}(2\pi, x, y)).$$

It is well known that $T$ is an area-preserving homeomorphism.

By applying the transform $x(t) = \gamma(t)\cos\theta(t)$, $y(t) = \gamma(t)\sin\theta(t)$ to (4.1), we get the equations for $\gamma(t)$ and $\theta(t)$,

$$\begin{cases} \dfrac{d\gamma}{dt} = \gamma\cos\theta\cdot\sin\theta - g(\gamma\cos\theta)\sin\theta + p(t)\sin\theta, \\ \dfrac{d\theta}{dt} = -\sin^2\theta - \dfrac{1}{\gamma}(g(\gamma\cos\theta)\cos\theta - p(t)\cos\theta), \end{cases} \tag{4.2}$$

whenever $\gamma \neq 0$.

Let $(\bar{\gamma}(t,\gamma,\theta), \bar{\theta}(t,\gamma,\theta))$ be the solution of (4.2) through the initial point $(\bar{\gamma}(0), \bar{\theta}(0)) = (\gamma,\theta)$. Then the map $T$ can also be written in the polar coordinate form

$$\gamma^* = \bar{\gamma}(2\pi,\gamma,\theta), \qquad \theta^* = \bar{\theta}(2\pi,\gamma,\theta) + 2l\pi, \tag{4.3}$$

where $l$ is an arbitrary integer. It can be easily seen that if $(\gamma,\theta)$ is such that

$$\bar{\gamma}(t,\gamma,\theta) > 0, \qquad t \in [0,2\pi], \tag{4.4}$$

then $\bar{\theta}(2\pi,\gamma,\theta)$ is well defined and continuous in $(\gamma,\theta)$, and moreover,

$$\bar{\theta}(2\pi,\gamma,\theta+2\pi) = \bar{\theta}(2\pi,\gamma,\theta) + 2\pi. \tag{4.5}$$

Next, we take the transform $z(t) = \rho(t)\cos\varphi(t)$, $w(t) = \rho(t)\sin\varphi(t)$ for system (2.1′). Then the resulting equations for $\rho(t)$ and $\varphi(t)$ are

$$\begin{cases} \dfrac{d\rho}{dt} = \rho\cos\varphi\cdot\sin\varphi - g(\rho\cos\varphi)\sin\varphi, \\ \dfrac{d\varphi}{dt} = -\sin^2\varphi - \dfrac{1}{\rho}g(\rho\cos\varphi)\cos\varphi. \end{cases} \tag{4.6}$$

Let $(\bar{\rho}(t,\rho,\varphi), \bar{\varphi}(t,\rho,\varphi))$ be the solution of (4.6) through the initial point $(\bar{\rho}(0), \bar{\varphi}(0)) = (\rho,\varphi)$.

LEMMA 4.1. *Let* $\Phi(\rho,\varphi) = \bar{\varphi}(2\pi,\rho,\varphi) - \varphi$, *and let* $\beta = \min\{2\pi, m\alpha A_1\}$, *where* $m$, $\alpha$ *and* $A_1$ *are given in* ($\mathrm{H}_3$) *and* (2.3). *Then we have*

$$\begin{cases} \Phi(\rho,\varphi) \le -2m\pi - \beta, & (\rho\cos\varphi, \rho\sin\varphi) \in \Gamma_{a_k}; \\ \Phi(\rho,\varphi) \ge -2m\pi + \beta, & (\rho\cos\varphi, \rho\sin\varphi) \in \Gamma_{b_k}. \end{cases} \tag{4.7}$$

PROOF. Let $(z,w) = (\rho\cos\varphi, \rho\sin\varphi) \in \Gamma_{a_k}$. Consider the solution $(\bar{\rho}(t,\rho,\varphi), \bar{\varphi}(t,\rho,\varphi))$ of (4.6). It follows from (2.3) and the second equation of (4.6) that

$$\bar{\varphi}'(t,\rho,\varphi) \le -A_1, \tag{4.8}$$

provided that $\rho$ is sufficiently large.

Since the solution $(\bar{z}(t,z,w), \bar{w}(t,z,w))$ of (2.1′) has the least period $\tau(a_k)$, we see that the time in which $\bar{\varphi}(t)$ has a decrement $2\pi$ is just $\tau(a_k)$. Write

$$\Phi(\rho,\varphi) = \bar{\varphi}(2\pi,\rho,\varphi) - \varphi = \bar{\varphi}(2\pi) - \bar{\varphi}(0) = -2l\pi - \sigma,$$

where $l \ge 0$ is an integer, and $0 \le \sigma < 2\pi$. Let $t_\sigma$ denote the time in which $\bar{\varphi}(t)$ decreases from $\varphi - 2l\pi$ to $\varphi - 2l\pi - \sigma$. Then we have

$$l\cdot\tau(a_k) + t_\sigma = 2\pi.$$

Since $0 \le t_\sigma < \tau(a_k)$, we obtain

$$2\pi = l \cdot \tau(a_k) + t_\sigma < (l+1)\tau(a_k) \le (l+1)\left(\frac{2\pi}{m} - \alpha\right).$$

It follows that $l \ge m$. If $l \ge m+1$, we have

$$\Phi(\rho, \varphi) \le -2l\pi \le -2(m+1)\pi. \tag{4.9}$$

Now, assume $l = m$. Then we have

$$t_\sigma = 2\pi - m \cdot \tau(a_k) \ge 2\pi - m\left(\frac{2\pi}{m} - \alpha\right) = m\alpha. \tag{4.10}$$

By (4.8) and (4.10), we obtain

$$-\sigma = \int_{l\cdot\tau(a_k)}^{l\cdot\tau(a_k)+t_\sigma} \bar{\varphi}'(t, \rho, \varphi)\, dt \le -A_1 t_\sigma \le -m\alpha A_1.$$

Thus we have

$$\Phi(\rho, \varphi) = -2l\pi - \sigma \le -2m\pi - m\alpha A_1. \tag{4.11}$$

Combining (4.9) and (4.11) yields the first inequality of (4.7).

The second inequality of (4.7) can be proved in a similar way. The proof of Lemma 4.1 is then completed.

LEMMA 4.2. *Let* $\Theta(\gamma, \theta) = \bar{\theta}(2\pi, \gamma, \theta) - \theta$. *Then there exists* $\gamma_0 > 0$ *such that, for* $\gamma \ge \gamma_0$,

$$|\Theta(\gamma, \theta) - \Phi(\gamma, \theta)| = |\bar{\theta}(2\pi, \gamma, \theta) - \bar{\varphi}(2\pi, \gamma, \theta)| < \beta. \tag{4.12}$$

PROOF. Let $(\bar{z}(t, x, y), \bar{w}(t, x, y))$ be the solution of (2.1′) through the initial point $(\bar{z}(0), \bar{w}(0)) = (x, y)$. Let

$$\begin{cases} u(t) = u(t, x, y) = \bar{x}(t, x, y) - \bar{z}(t, x, y), \\ v(t) = v(t, x, y) = \bar{y}(t, x, y) - \bar{w}(t, x, y). \end{cases}$$

Then we have

$$\frac{du}{dt} = v, \qquad \frac{dv}{dt} = p(t) - g'(\sigma(t))u,$$

where $\sigma(t) = \bar{x}(t) + \lambda(t)(\bar{z}(t) - \bar{x}(t))$, $0 \le \lambda(t) \le 1$.

Let $\eta(t) = (u^2(t) + v^2(t))^{1/2}$. Then we have

$$\eta \frac{d\eta}{dt} = uv + p(t)v - g'(\sigma(t))uv.$$

It follows from $(\mathrm{H}_1)$ that

$$\left|\frac{d\eta}{dt}\right| \le \frac{1}{2}(1+K)\eta + B, \tag{4.13}$$

where $B$ is a bound of $|p(t)|$ in $[0, 2\pi]$. The differential inequality (4.13) together with $\eta(0) = 0$ yields

$$\eta(t) \le \frac{2B}{K+1}[e^{(K+1)\pi} - 1] \equiv H_0, \tag{4.14}$$

for $t \in [0, 2\pi]$.

Next write $\psi(t) = \psi(t,\gamma,\theta) = \bar{\theta}(t,\gamma,\theta) - \bar{\varphi}(t,\gamma,\theta)$, where $(\gamma,\theta)$ is the polar coordinate of $(x,y)$, i.e., $(\gamma\cos\theta, \gamma\sin\theta) = (x,y)$. It is clear that if $|\psi(t)| < \pi$, then $\psi(t)$ is just the angle between the vectors $(\bar{x}(t), \bar{y}(t))$ and $(\bar{z}(t), \bar{w}(t))$. By the law of cosines, we have

$$\cos\psi(t) = \frac{\bar{\gamma}^2(t) + \bar{\rho}^2(t) - \eta^2(t)}{2\bar{\gamma}(t)\bar{\rho}(t)} \geq 1 - \frac{H_0^2}{2\bar{\gamma}(t)\bar{\rho}(t)}.$$

On the other hand, we have $\bar{\gamma}(t) \geq \bar{\rho}(t) - \eta(t) \geq \bar{\rho}(t) - H_0$. Therefore, under the assumption that $|\psi(t)| < \pi$ and $\bar{\rho}(t) - H_0 > 0$, we have

$$\cos\psi(t) \geq 1 - \frac{H_0^2}{2\bar{\rho}(t)(\bar{\rho}(t) - H_0)}. \tag{4.15}$$

Note that

$$F(\gamma,\varphi) = \inf_{0\leq t\leq 2\pi} \bar{\rho}(t) = \inf_{0\leq t\leq 2\pi} \bar{\rho}(t,\gamma,\varphi)$$

becomes arbitrarily large if $\gamma$ is sufficiently large. It follows that there is a constant $\gamma_0 > 0$ such that, for $\gamma \geq \gamma_0$ and $t \in [0, 2\pi]$,

$$\bar{\rho}(t) - H_0 > 0; \qquad \frac{H_0^2}{2\bar{\rho}(t)(\bar{\rho}(t) - H_0)} < 1 - \cos\delta, \tag{4.16}$$

where $\delta = \min(\pi/2, \beta)$. From (4.15) and (4.16), we conclude that if $|\psi(t)| < \pi$ and $t \in [0, 2\pi]$, then the inequality

$$|\psi(t)| < \delta \tag{4.17}$$

holds.

Since $\psi(0) = 0$ and $\psi(t)$ varies continuously as $t$ increases from 0 to $2\pi$, we can see that (4.17) holds for any $t \in [0, 2\pi]$. In particular, we have

$$|\psi(2\pi)| = |\bar{\theta}(2\pi,\gamma,\theta) - \bar{\varphi}(2\pi,\gamma,\theta)| < \delta \leq \beta,$$

for $\gamma \geq \gamma_0$. This proves Lemma 4.2.

PROOF OF THEOREM 2.1. Let $c_1 \geq c_0$ be so large that $(\gamma,\theta) \in \Gamma_c$ with $c \geq c_1$ implies $\gamma \geq \gamma_0$, where $\gamma_0$ is specified in Lemma 4.2. There is no loss of generality to assume $a_k \geq c_1$ for $k \geq n_0$. It follows that $\bar{\gamma}(t,\gamma,\theta) \geq \bar{\rho}(t) - H_0 > 0$, $t \in [0, 2\pi]$, provided that $(\gamma,\theta) \in \mathcal{A}_k$ for $k \geq n_0$.

Thus the restriction $T|\mathcal{A}_k$ can be written in (4.3), where the function $\bar{\theta}(2\pi,\gamma,\theta)$ is continuous on $\mathcal{A}_k$ and satisfies the relation (4.4). Now, we put the integer $l = m$ in (4.3). Then (4.3) can be rewritten in the form of

$$\gamma^* = \bar{\gamma}(2\pi,\gamma,\theta), \qquad \theta^* = \theta + \Theta_1(\gamma,\theta),$$

with $\Theta_1(\gamma,\theta) = \Theta(\gamma,\theta) + 2m\pi$. By Lemma 4.2, we obtain

$$|\Theta_1(\gamma,\theta) - \Phi(\gamma,\theta) - 2m\pi| < \beta,$$

which together with (4.7) yields

$$\begin{aligned} \Theta_1(\gamma,\theta) &< 0, \qquad (\gamma\cos\theta, \gamma\sin\theta) \in \Gamma_{a_k}; \\ \Theta_1(\gamma,\theta) &> 0, \qquad (\gamma\cos\theta, \gamma\sin\theta) \in \Gamma_{b_k}, \end{aligned}$$

for $k \geq n_0$, where $n_0$ is large enough.

This proves the validity of condition (iii) of Theorem A for the restriction $T|\mathcal{A}_k$

($k \geq n_0$). Since $\gamma \geq \gamma_0$ implies $\bar{\gamma}(2\pi, \gamma, \theta) \geq \bar{\rho}(2\pi) - H_0 > 0$, so the condition (ii) of Theorem A can be easily verified. By Lemma 2.1, the condition (i) of Theorem A also holds. Therefore, we can apply Theorem A to ensure the existence of at least two fixed points of $T$ in $\mathcal{A}_k$ ($k \geq n_0$). This means that (1.1) has at least two $2\pi$-periodic solutions with initial points in $\mathcal{A}_k$. In the same way, we can prove that $T$ has at least two fixed points in $\mathcal{B}_k$ which correspond to two $2\pi$-periodic solutions of (1.1). Since each periodic solution of (1.1) is bounded by a $\Gamma_c$, so the above-specified $2\pi$-periodic solutions of (1.1) constitute an infinite class.

The proof of Theorem 2.1 is thus completed.

**5.** Finally, we give a concrete example for applications of Theorem 2.1.

Let $m$ be a positive integer, and let

$$\bar{g}(x) = x\left[m^2 + \frac{1}{10}\cos\log(1+x^2) - \frac{x^2}{10(1+x^2)}\sin\log(1+x^2)\right].$$

A direct calculation yields

$$m^2 - \tfrac{1}{2} \leq \bar{g}'(x) \leq m^2 + \tfrac{1}{2}.$$

It follows that ($H_1$) and ($H_2$) hold for $\bar{g}(x)$.

Consider the auxiliary equation

$$\frac{d^2 z}{dt^2} + \bar{g}(z) = 0. \tag{5.1}$$

According to (2.2), we have

$$V(z, w) \equiv \tfrac{1}{2}w^2 + \bar{G}(z) = c,$$

where

$$\bar{G}(z) = \int_0^z \bar{g}(u)\,du = \frac{1}{2}z^2\left[m^2 + \frac{1}{10}\cos\log(1+z^2)\right].$$

Take $h = h(c) > 0$ such that $\bar{G}(h(c)) = c$. Since $\bar{G}(z)$ is an even function, we have

$$\tau(c) = 2\sqrt{2}\int_0^{h(c)} \frac{du}{\sqrt{c - \bar{G}(u)}} = \frac{4}{\sqrt{Q(h)}}\int_0^1 \frac{d\xi}{\sqrt{1 - \xi^2 Q(h\xi)/Q(h)}}$$

where

$$Q(x) = m^2 + \frac{1}{10}\cos\log(1+x^2).$$

Let $\alpha_k = \sqrt{e^{2k\pi} - 1}$, $\beta_k = \sqrt{e^{(2k+1)\pi} - 1}$, $a_k = \bar{G}(\alpha_k)$ and $b_k = \bar{G}(\beta_k)$. Then

$$a_k < b_k \qquad (k = 1, 2, \dots),$$

and

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = \infty.$$

Since the inequalities

$$0 < \frac{Q(\alpha_k \xi)}{Q(\alpha_k)} \leq 1, \qquad \frac{Q(\beta_k \xi)}{Q(\beta_k)} \geq 1$$

hold for $\xi \in [0,1]$, we have

$$\begin{aligned}\tau(a_k) &= \frac{4}{\sqrt{Q(\alpha_k)}}\int_0^1 \frac{d\xi}{\sqrt{1-\xi^2 Q(\alpha_k\xi)/Q(\alpha_k)}} \\ &\le \frac{4\sqrt{10}}{\sqrt{10m^2+1}}\int_0^1 \frac{d\xi}{\sqrt{1-\xi^2}} \le \frac{2\pi}{m} - \bar{\alpha},\end{aligned}$$

and

$$\begin{aligned}\tau(b_k) &= \frac{4}{\sqrt{Q(\beta_k)}}\int_0^1 \frac{d\xi}{\sqrt{1-\xi^2 Q(\beta_k\xi)/Q(\beta_k)}} \\ &\ge \frac{4\sqrt{10}}{\sqrt{10m^2-1}}\int_0^1 \frac{d\xi}{\sqrt{1-\xi^2}} \ge \frac{2\pi}{m} + \bar{\alpha},\end{aligned}$$

where

$$\bar{\alpha} = \frac{2\pi}{m}\cdot\min\left\{1 - \frac{\sqrt{10}m}{\sqrt{10m^2+1}}, \frac{\sqrt{10}m}{\sqrt{10m^2-1}} - 1\right\}.$$

This proves that $(H_3)$ is also valid for $\bar{g}(x)$.

Therefore, the function $\bar{g}(x)$ satisfies all the assumptions of Theorem 2.1, and the corresponding equation

$$\frac{d^2 x}{dt^2} + \bar{g}(x) = p(t)$$

has infinitely many $2\pi$-periodic solutions for any $2\pi$-periodic function $p(t) \in C(R,R)$.

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