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COMPLETELY BOUNDED MAPS ON C^*-ALGEBRAS AND INVARIANT OPERATOR RANGES

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ABSTRACT. We give a new proof that every completely bounded map from a C^*-algebra into $\mathfrak{L}(\mathcal{H})$ lies in the linear span of the completely positive maps. In addition, we obtain an equivalent reformulation of the invariant operator range problem.

1. Introduction. A question of Kadison [**8**] asks whether or not every bounded homomorphism from a C^*-algebra into the algebra of operators on a Hilbert space, $\mathfrak{L}(\mathcal{H})$, is similar to a *-homomorphism. Hadwin [**7**] has shown that a bounded unital homomorphism from a C^*-algebra into $\mathfrak{L}(\mathcal{H})$ is similar to a *-homomorphism if and only if the homomorphism belongs to the span of the completely positive maps. Recently, Wittstock [**9**, Satz 4.5] proved that the span of the completely positive maps from a C^*-algebra into $\mathfrak{L}(\mathcal{H})$ is identical with the set of completely bounded maps. Together these two results prove that a bounded unital homomorphism from a C^*-algebra into $\mathfrak{L}(\mathcal{H})$ is similar to a *-homomorphism if and only if it is completely bounded (see [**6**, Theorem 1.10] for another proof, and the following Remark). Because of a well-known connection between the similarity question and the question of when derivations into $\mathfrak{L}(\mathcal{H})$ are inner [**2**, **3**] these results also lead to the identification of the set of inner derivations with the set of completely bounded derivations. The similarity question also has connections with the invariant operator range question, which we shall discuss in §3.

While Hadwin's proof uses only well-known results, Wittstock's proof rests on some recent deep results on W^*-algebras and uses a generalized Hahn-Banach Theorem for set-valued matrix sublinear functionals. Because of the many implications of this pair of theorems, we felt that an elementary proof of the portion of Wittstock's theory necessary for the above results would be of some value. The purpose of this note is to present such a proof, which involves some, perhaps new, observations about completely positive maps and uses only Arveson's extension theorem [**1**, Theorem 1.2.9].

In addition, we use these results to give an equivalent reformulation of the invariant operator range problem.

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2. Wittstock's Theorem. In this section we give an elementary proof of Wittstock's Theorem which uses only Arveson's extension theorem for completely contractive maps.

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, let $\phi\colon \mathcal{A} \to \mathcal{B}$ be a bounded linear map and let $\phi_n = \phi \otimes 1_n\colon \mathcal{A} \otimes M_n \to \mathcal{B} \otimes M_n$. We call ϕ *n-positive* (respectively, *n-contractive*) whenever ϕ_n is positive (respectively, $\|\phi_n\| \leqslant 1$). We call ϕ *completely positive* (respectively, *completely contractive*) if ϕ is n-positive (respectively, n-contractive) for all n. Similarly, if $\sup_n \|\phi_n\|$ is finite, then we call ϕ *completely bounded* and set $\|\phi\|_{cb} = \sup_n \|\phi_n\|$.

We shall use $\mathcal{A}^h$ to denote the self-adjoint elements of $\mathcal{A}$. For $\mathcal{S} \subseteq \mathcal{A}^h$ a real linear subspace, we let $\tilde{\mathcal{S}} = \mathcal{S} + i\mathcal{S}$ denote its complex span. If $\phi\colon \mathcal{S} \to \mathcal{B}^h$ is real linear, then we define $\tilde{\phi}\colon \tilde{\mathcal{S}} \to \mathcal{B}$ by $\tilde{\phi}(S_1 + iS_2) = \phi(S_1) + i\phi(S_2)$. We shall call ϕ a *real n-contraction* whenever $\tilde{\phi}_n\colon (\tilde{\mathcal{S}} \otimes M_n)^h \to (\mathcal{B} \otimes M_n)^h$ is a contraction, and call ϕ a *real complete contraction* if it is a real n-contraction for all n.

PROPOSITION 2.1. *If $\mathcal{S} \subseteq \mathcal{A}^h$ is a real linear subspace, and $\phi\colon \mathcal{S} \to \mathcal{B}^h$ is a real $2n$-contraction, then $\tilde{\phi}\colon \tilde{\mathcal{S}} \to \mathcal{B}$ is an n-contraction. In particular, if ϕ is a real complete contraction, then $\tilde{\phi}$ is a complete contraction.*

PROOF. Let $T \in \tilde{\mathcal{S}} \otimes M_n$. One has

$$\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix} \in \left(\tilde{\mathcal{S}} \otimes M_{2n}\right)^h$$

and

$$\|\tilde{\phi}_n(T)\| = \left\|\tilde{\phi}_{2n}\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}\right\| \leqslant \left\|\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}\right\| = \|T\|. \quad \square$$

The following lemma is a portion of [**9**, Satz 4.2], we include the proof for completeness. Recall that if $L\colon \mathcal{A} \to \mathcal{B}$ then we define L^*, by $L^*(a) = L(a^*)^*$. Throughout this paper we identify a matrix H in M_n with $1 \otimes H$ in $\mathbf{C} \otimes M_n$.

LEMMA 2.2. *Let $L\colon \mathcal{A} \to \mathcal{B}$ be completely contractive, with $L = L^*$. If $A \in (\mathcal{A} \otimes M_n)^h$, $H \in M_n^h$ and $H \pm A \geqslant 0$, then $H \pm L_n(A) \geqslant 0$.*

PROOF. Since $\pm A \leqslant H$ if and only if $\pm A \leqslant H + \gamma I_n$ for all positive real γ, it suffices to assume H is positive and invertible. But in this case, one has $\pm H^{-1/2}AH^{-1/2} \leqslant I_n$ so that $\|H^{-1/2}AH^{-1/2}\| \leqslant 1$ and hence $\pm L_n(H^{-1/2}AH^{-1/2}) \leqslant I_n$, since it is a Hermitian of norm less than one. Finally, note that $L_n(H^{-1/2}AH^{-1/2}) = H^{-1/2}L_n(A)H^{-1/2}$, from which the result follows. $\square$

We shall now show that applying complete contractions off-diagonally yields complete contractions. We let $\mathcal{A}_1$ denote the smallest unital C^*-algebra containing $\mathcal{A}$ (so $\mathcal{A}_1 = \mathcal{A}$ if $\mathcal{A}$ has a unit).

THEOREM 2.3. *Let $L\colon \mathcal{A} \to \mathcal{B}$ be a complete contraction with $L = L^*$ and let*

$$\mathcal{S} = \left\{\begin{pmatrix} \gamma & a \\ a^* & s \end{pmatrix} \mid \gamma, s \in \mathbf{R}, a \in \mathcal{A}\right\} \subseteq (\mathcal{A}_1 \otimes M_2)^h.$$

Then the map $\phi\colon \mathfrak{S} \to (\mathfrak{B}_1 \otimes M_2)^h$ *defined by*

$$\phi\begin{pmatrix} \gamma & a \\ a^* & s \end{pmatrix} = \begin{pmatrix} \gamma & L(a) \\ L(a^*) & s \end{pmatrix}$$

is a real complete contraction.

PROOF. An element of $(\tilde{\mathfrak{S}} \otimes M_n)^h$, after rewriting in the canonical fashion as a 2×2 matrix of $n \times n$ blocks, is of the form

$$\begin{pmatrix} H & A \\ A^* & K \end{pmatrix},$$

with $A \in (\mathfrak{A} \otimes \mathrm{M}_n)^h$, $H, K \in M_n^h$. Furthermore, we have that

$$\tilde{\phi}_n\begin{pmatrix} H & A \\ A^* & K \end{pmatrix} = \begin{pmatrix} H & L_n(A) \\ L_n(A^*) & K \end{pmatrix}.$$

Conjugation by the matrix

$$\begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}$$

shows that

$$\left\|\begin{pmatrix} H & A \\ A^* & K \end{pmatrix}\right\| = \left\|\begin{pmatrix} H & -A \\ -A^* & K \end{pmatrix}\right\|$$

and hence,

$$\begin{aligned} \left\|\begin{pmatrix} H & A \\ A^* & K \end{pmatrix}\right\| &= \inf\left\{\gamma \mid \gamma \cdot I_{2n} \pm \begin{pmatrix} H & A \\ A^* & K \end{pmatrix} \geqslant 0\right\} \\ &= \inf\left\{\gamma \mid \gamma \cdot I_{2n} \pm \begin{pmatrix} H & 0 \\ 0 & K \end{pmatrix} \pm \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \geqslant 0\right\}. \end{aligned}$$

By Lemma 2.2, if

$$\left[\gamma \cdot I_{2n} \pm \begin{pmatrix} H & 0 \\ 0 & K \end{pmatrix}\right] \pm \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \geqslant 0,$$

then

$$\left[\gamma \cdot I_{2n} \pm \begin{pmatrix} H & 0 \\ 0 & K \end{pmatrix}\right] \pm \begin{pmatrix} 0 & L_n(A) \\ L_n(A^*) & 0 \end{pmatrix} \geqslant 0.$$

Thus,

$$\begin{aligned} \left\|\begin{pmatrix} H & A \\ A^* & K \end{pmatrix}\right\| &\geqslant \inf\left\{\gamma \mid \gamma \cdot I_{2n} \pm \begin{pmatrix} H & 0 \\ 0 & K \end{pmatrix} \pm \begin{pmatrix} 0 & L_n(A) \\ L_n(A^*) & 0 \end{pmatrix} \geqslant 0\right\} \\ &= \left\|\begin{pmatrix} H & L_n(A) \\ L_n(A^*) & K \end{pmatrix}\right\|, \end{aligned}$$

from which the result follows. □

Noticing that the complex span of $\mathcal{S}$,

$$\tilde{\mathcal{S}} = \left\{ \begin{pmatrix} \lambda & a \\ b & \mu \end{pmatrix} \mid \lambda, \mu \in \mathbf{C}, \text{ and } a, b \in \mathcal{A} \right\}$$

and applying Proposition 2.1 we have

COROLLARY 2.4. *Let L: $\mathcal{A} \to \mathcal{B}$ be a complete contraction with $L = L^*$, then for any $X, Y \in M_n$, and $A, B \in \mathcal{A} \otimes M_n$ we have that*

$$\left\| \begin{pmatrix} X & A \\ B & Y \end{pmatrix} \right\| \geq \left\| \begin{pmatrix} X & L_n(A) \\ L_n(B) & Y \end{pmatrix} \right\|.$$

Wittstock's Theorem now follows by some algebraic manipulations, which we separate into the following.

THEOREM 2.5. *Let L: $\mathcal{A} \to \mathcal{B}$ be a complete contraction with $L = L^*$. If the complete real contraction ϕ of Theorem 2.3 possesses an extension to a completely positive map,*

$$\Phi: \mathcal{A}_1 \otimes M_2 \to \mathcal{B}_1 \otimes M_2,$$

then there exists a unital completely positive map ψ: $\mathcal{A}_1 \to \mathcal{B}_1$ such that the maps $\psi \pm L$: $\mathcal{A} \to \mathcal{B}_1$ are completely positive.

PROOF. If $a \in \mathcal{A}_1^h$ and $a \leq \gamma$ for some real γ, then

$$\Phi\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \leq \Phi\begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that there exists ϕ_1: $\mathcal{A}_1 \to \mathcal{B}_1$ such that,

$$\Phi\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \phi_1(a) & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for all } a \in \mathcal{A}_1.$$

Similar manipulations show that Φ must have the following form,

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \phi_1(a) & L(b) \\ L(c) & \phi_2(d) \end{pmatrix} \quad \text{for } a, d \in \mathcal{A}_1,$$

and $b, c \in \mathcal{A}$. Furthermore, ϕ_1 and ϕ_2 must be unital, and completely positive.

Note that the maps γ: $\mathcal{A}_1 \to \mathcal{A}_1 \otimes M_2$ given by $\gamma(a) = \left(\begin{smallmatrix} a & a \\ a & a \end{smallmatrix}\right)$ and δ: $\mathcal{B}_1 \otimes M_2 \to \mathcal{B}_1$ given by

$$\delta\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^* \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = b_1 + b_2 + b_3 + b_4$$

are completely positive. Thus, $\delta \circ \Phi \circ \gamma(a) = \phi_1(a) + \phi_2(a) + 2L(a)$ for $a \in \mathcal{A}$ is completely positive. Setting $\psi = \frac{1}{2}(\phi_1 + \phi_2)$, we have that ψ is unital, completely positive, $\|\psi\| = 1$ and that $\psi + L$: $\mathcal{A} \to \mathcal{B}_1$ is completely positive.

If we first conjugate Φ by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and then compose with δ and γ, we find that $\psi - L$ is completely positive. □

The following is the special case of Wittstock's Theorem [**9**, Satz 4.5] with $\mathcal{C} = \mathbf{C}$, $\mathcal{B} = \mathcal{L}(\mathcal{H})$.

COROLLARY 2.6. *Let L: $\mathcal{A} \to \mathcal{L}(\mathcal{H})$ be completely bounded, with $L = L^*$, then there exists a completely positive map ψ: $\mathcal{A} \to \mathcal{L}(\mathcal{H})$, $\|\psi\| \leqslant \|L\|_{cb}$ such that $\psi \pm L$ are completely positive maps. Furthermore*

$$L = \tfrac{1}{2}(\psi + L) - \tfrac{1}{2}(\psi - L)$$

expresses L as a difference of completely positive maps each of which has norm less than $\|L\|_{cb}$.

PROOF. Dividing L by $\|L\|_{cb}$ reduces us to the case of a complete contraction. By Arveson's extension theorem the desired completely positive extension Φ of Theorem 2.5 exists. □

REMARK 2.7. More generally, if $\mathcal{B}$ is any unital, injective C^*-algebra the desired extension Φ exists.

The canonical decomposition of a linear map into a self-adjoint and a skew-adjoint part, shows that every completely bounded map into $\mathcal{L}(\mathcal{H})$ is in the span of the completely positive maps. Conversely, any map in the span of the completely positive maps is easily seen to be completely bounded.

We do not know when a complete contraction on a nonunital C^*-algebra can be extended to a complete contraction on the algebra with unit adjoined. When the range algebra is $\mathcal{L}(\mathcal{H})$ it is not difficult to show that this can be accomplished. When the completely positive extension Φ of Theorem 2.5 exists then the $(1,2)$-corner of Φ yields a complete contraction from $\mathcal{A}_1$ to $\mathcal{B}_1$ which extends the original complete contraction.

3. Invariant operator ranges. Let $\mathcal{A}$ be a unital C^*-subalgebra of $\mathcal{L}(\mathcal{H})$, and let $T \in \mathcal{L}(\mathcal{H})$ be such that the range of T, $\mathcal{R}(T)$, is an invariant linear manifold for $\mathcal{A}$. The invariant operator range problem asks if there exists a bounded operator S which commutes with $\mathcal{A}$ such that $\mathcal{R}(T) = \mathcal{R}(S)$. In this section, we shall prove that the invariant operator range problem is equivalent to the range of the infinite-ampliation of T being an invariant subspace for the tensor product of $\mathcal{A}$ with the compacts. Our proof is based on a well-known connection of this problem with the similarity problem, originally due to Foias [5], which we shall outline below.

Let $\mathcal{A}$ and T be as above and let $\mathcal{M}$ be the orthocomplement of the kernel of T. For $A \in \mathcal{A}$ and for $x \in \mathcal{M}$ there exists a unique $y \in \mathcal{M}$ such that $ATx = Ty$, and we set $\rho(A)x = y$. An application of the closed graph theorem shows that $\rho(A) \in \mathcal{L}(\mathcal{M})$ and a second application shows that ρ: $\mathcal{A} \to \mathcal{L}(\mathcal{M})$ is a continuous homomorphism. By [5], if ρ is similar to a *-homomorphism, then there exists an operator S in the commutant of $\mathcal{A}$ such that $R(S) = R(T)$.

Let $\mathcal{K}$ denote the compact operators on an infinite dimensional separable Hilbert space $\mathcal{H}'$, and let the C^*-algebra $\mathcal{A} \otimes \mathcal{K}$ be represented on $\mathcal{H} \otimes \mathcal{H}'$ spatially.

THEOREM 3.1. *Let $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ be a C^*-subalgebra, containing $1_{\mathcal{H}}$, and let $T \in \mathcal{B}(\mathcal{H})$ be such that $\mathcal{R}(T)$ is an invariant linear manifold for $\mathcal{A}$. A necessary and sufficient condition that there exists an operator in the commutant of $\mathcal{A}$ with $\mathcal{R}(S) = \mathcal{R}(T)$ is that $\mathcal{R}(T \otimes 1_{\mathcal{H}'})$ be an invariant linear manifold for $\mathcal{A} \otimes \mathcal{K}$.*

PROOF. To see necessity, let S be in the commutant of $\mathcal{A}$ with $\mathcal{R}(S) = \mathcal{R}(T)$. An application of the closed graph theorem shows that $\mathcal{R}(S \otimes 1_{\mathcal{K}'}) = \mathcal{R}(T \otimes 1_{\mathcal{K}'})$, so it will suffice to show that $\mathcal{R}(S \otimes 1_{\mathcal{K}'})$ is invariant for $\mathcal{A} \otimes \mathcal{K}$. But this is immediate since $S \otimes 1_{\mathcal{K}'}$ commutes with $\mathcal{A} \otimes \mathcal{K}$.

For sufficiency, first observe that if $\mathcal{M}$ is the orthocomplement of the kernel of T, then $\mathcal{M} \otimes \mathcal{K}'$ is the orthocomplement of the kernel of $T \otimes 1_{\mathcal{K}'}$. Thus, as in the remarks proceeding the theorem, we may for $B \in \mathcal{A} \otimes \mathcal{K}$ define $\rho'(B) \in \mathcal{L}(\mathcal{M} \otimes \mathcal{K}')$ by $\rho'(B)x = y$ where $B(T \otimes 1_{\mathcal{K}'})x = (T \otimes 1_{\mathcal{K}'})y$. In this fashion we obtain a continuous homomorphism, $\rho'\colon \mathcal{A} \otimes \mathcal{K} \to \mathcal{L}(\mathcal{M} \otimes \mathcal{K}')$.

We claim that the existence of the homomorphism ρ' implies the complete boundedness of the homomorphism, $\rho\colon \mathcal{A} \to \mathcal{L}(\mathcal{M})$, described in the remarks preceeding the theorem. Note first, that if $A \otimes K$ is an elementary tensor, then $\rho'(A \otimes K) = \rho(A) \otimes K$ where the second tensor product is the spatial one. This follows directly from the definition of $\rho'(A \otimes K)$.

If we embed M_n as a C^*-subalgebra of $\mathcal{K}$, this yields an embedding of $\mathcal{A} \otimes M_n$ as a C^*-subalgebra of $\mathcal{A} \otimes \mathcal{K}$. Since every element $B \in \mathcal{A} \otimes M_n$ can be represented as a finite linear combination of elementary tensors, say $B = \sum_i A_i \otimes K_i$, we have

$$\rho'(B) = \sum_i \rho(A_i) \otimes K_i.$$

Hence, $\|\rho \otimes 1_n(B)\| = \|\rho'(B)\|$ so that $\|\rho \otimes 1_n\| \leqslant \|\rho'\|$, that is, ρ is completely bounded. Applying the result referred to earlier [**6**, Theorem 1.10], that ρ being completely bounded is equivalent to ρ being similar to a *-homomorphism, completes the proof. □

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