RECOVERABILITY OF SOME CLASSES OF ANALYTIC FUNCTIONS FROM THEIR BOUNDARY VALUES

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Abstract. The technique devised by D. J. Patil to recover the functions of the Hardy space $H^p (1 \leq p < \infty)$ from the restrictions of their boundary values to a set of positive measure on the unit circle was modified by S. E. Zarantonello in order to extend the result to $H^p (0 < p < 1)$. In this paper, we show that Zarantonello's technique can be slightly modified to extend the result to a larger class of analytic functions in the unit disc. In particular, if $f(z)$ is analytic in the unit disc and satisfies

$$\lim_{r \to 1} (1 - r)^\beta \log M(r, f) = 0 \quad \text{for some } \beta > 1,$$

then $f(z)$ can be recovered from the restriction of its boundary value to an open arc.

1. Introduction. Let $\mathbb{D}$ denote the open unit disc, $\partial \mathbb{D}$ its boundary, i.e., the unit circle, and $\mu$ the normalized Lebesgue measure on $\partial \mathbb{D}$. Furthermore, let $H^p (0 < p \leq \infty)$ denote the Hardy class of analytic functions in $\mathbb{D}$. It is well known that if $f(z) \in H^p$ and $\lim_{r \to 1} f(re^{i\theta}) = 0$ on a set $E \subset \partial \mathbb{D}$ of positive measure, then $f(z)$ is identically zero. We call this property the uniqueness property. In a sense, this means that functions in $H^p$ are uniquely determined by their values on $E$.

The question of whether a function $f \in H^p$ can be recovered from its restriction to $E$ was answered in the affirmative by D. Patil [1] for $1 \leq p < \infty$, who constructed a sequence of analytic functions that converged to $f$ uniformly on compact subsets of $\mathbb{D}$ as well as in the norm. Modifying Patil's technique, S. Zarantonello [4] and G. Walker [3] independently were able to extend the results to $0 < p < 1$ but in a slightly more restrictive form.

A function $f(z) \in H^p (0 < p < 1)$ can be recovered from the restriction of its distributional boundary value to $E$ where $E$ now is an open arc in $\partial \mathbb{D}$.

Since there are larger classes of analytic functions in $\mathbb{D}$ having the uniqueness property, e.g., the Nevanlinna class $N$, a natural question immediately arises, can functions in these classes be recovered from their restrictions to $E$? The purpose of this article is to show that the answer is "yes" for a large class of analytic functions. More precisely, if we denote by $\mathcal{F}(\alpha) (0 < \alpha < 1)$ the class of all analytic functions in $\mathbb{D}$ having the property

$$\int_0^1 M(r, f) \exp(-c/(1 - r)^\beta) \, dr < \infty \text{ for all } c > 0$$

where...
But \((k + n)^a \leq k^a + n^a \leq 2(k + n)^a\), therefore

\[
I_2 \leq \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{\infty} |b_k| \exp\left(-\frac{1}{2}c |k|^a - \frac{1}{2}c |n|^a\right) \leq B \|f\|_{c/2}
\]

where \(B = \sum_{k=0}^{\infty} |b_k| \exp(-\frac{1}{2}c |k|^a)\) which certainly converges since \(|b_k| \leq Ae^{-b|k|^a}\) for some \(b > 0\). On the other hand,

\[
I_1 \leq \sum_{n=0}^{\infty} \sum_{k=-n}^{n} A e^{-b|k|^a - c|k+n|^a} 
\]

(3.3)

\[
\leq A \sum_{n=0}^{\infty} |a_n| ne^{-2dn^a} \leq \tilde{B} \sum_{n=0}^{\infty} |a_n| e^{-dn^a} = \tilde{B} \|f\|_{d}
\]

where \(d = \frac{1}{2} \min(b, c)\) and \(\tilde{B} = \max_{0<n} A(n/e^{dn^a}) < \infty\).

From (3.1), (3.2) and (3.3), it follows that \(\|S_f\|_c \leq K \|f\|_{c/2}\) where \(K = \max(B, \tilde{B})\) and \(\tilde{c} = \min(c/2, d)\).

(ii) Let \(f \in \mathcal{H}(\alpha)\) and \(F(z)\) be its holomorphic extension to \(\bar{D}\). From Corollaries 3.1 and 3.2 of [5], it follows that \(F_r(w) = F(rw) \to f\) in \(\mathcal{H}(\alpha)\) and that \(H^2\) is dense in \(\mathcal{H}(\alpha)\). Moreover, \(F_r(w)\) is the boundary value of a function in \(H^2\), namely \(F_r(z)\). Now an argument similar to that of Theorem 3.2 of [4] finishes the proof.

4. Recoverability Theorem. In this section we shall use the same notation as [4]. Let \(E\) be an open arc in \(\partial D\) and \(\psi \in \mathcal{O}_a\) such that \(0 < \psi < 1\), supp \(\psi \subset E\) and \(\mu(J) > 0\) where \(J = \{w \in \partial D : \psi(w) = 1\}\). Such a \(\psi\) certainly exists since the class \(\mathcal{O}_a\) is nonquasianalytic. For each \(0 < \lambda < 1\) define

\[
\phi_\lambda(w) = \frac{1}{1 - \lambda \psi(w)}
\]

and

\[
H_\lambda(z) = \exp\left\{\frac{1}{2} \int_{\partial D} \frac{w + z}{w - z} \log \phi_\lambda(w) d\mu\right\}, \quad z \in D.
\]

As is shown in [1 and 4], if we denote the boundary value of \(H_\lambda(z)\) by \(h_\lambda(w)\), we have

(a) \(|h_\lambda(w)|^{-2} = \phi_\lambda(w)\).

(b) \(h_\lambda\) and \(h_\lambda^{-1}\) are in \(H^\infty(\partial D)\).

**LEMMA 2.** (i) \(\phi_\lambda \in \mathcal{O}_a\).

(ii) \(S_{\phi_\lambda}: \mathcal{O}_a^* \to \mathcal{O}_a^*\) is invertible with inverse \(S_{\phi_\lambda}^{-1} = S_{h_\lambda} S_{\bar{h}_\lambda}\).

(iii) For any \(c > 0\), there exists \(\tilde{c}\) such that

\[
\|S_{\phi_\lambda}^{-1}f\|_c \leq K \|f\|_\tilde{c}\quad \text{where } K \text{ is independent of } \lambda.
\]

**PROOF.** (i) Since \(\psi \in \mathcal{O}_a\) and \(\mathcal{O}_a\) is an inverse-closed, nonquasianalytic class of functions [2], it follows immediately that \(\phi_\lambda \in \mathcal{O}_a\).

(ii) It suffices to show that \(h_\lambda \in \mathcal{O}_a\). Let \(\bar{h}_\lambda(w) = e^{-(u(w) + i\phi(w))}\). Since \(\log \phi_\lambda(w) \in \mathcal{O}_a\) by part (i) and Theorem A of [2], it is easy to see that \(u(w)\) and \(\phi(w)\) are also in \(\mathcal{O}_a\). We invoke Theorem A once more to show that \(h_\lambda\) is indeed in \(\mathcal{O}_a\).

(iii) \(\|S_{h_\lambda} S_{\bar{h}_\lambda} f\|_c \leq \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\bar{h}_\lambda(k)| \|\bar{h}_\lambda(m)| \|\bar{f}(n)| e^{-c|k-m+n|^a}\).
Since $|h_\lambda(w)| \leq (\phi_\lambda(w))^{-1/2} \leq 1$, it follows that the sequence \( \{h_\lambda(k)\} \) is uniformly bounded. Now an argument similar to the one given in Lemma 1 yields the result.

**Lemma 3.** For fixed \( z \in \mathbb{D} \) and \( c > 0 \), \( \lim_{\lambda \to 1} \| S^{-1}_\phi C_z \|_c = 0 \).

**Proof.** The relation (cf. [4, equation (4.4.2)])

\[
|H_\lambda(z)| = \exp \left\{ -\frac{1}{2} \int \frac{1 - |z|^2}{1 - \overline{z}w} \log \phi_\lambda(w) \, d\mu(w) \right\} \leq (1 - \lambda)^a
\]

where \( 2\alpha = \frac{(1 - |z|)}{1 + |z|} \mu(J) > 0 \), shows that \( H_\lambda(z) \to 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( \lambda \to 1 \). Since \( H_\lambda(z) = \sum_{k=0}^\infty \hat{h}_\lambda(k) z^k \), it is routine to show that \( \hat{h}_\lambda(k) \to 0 \) as \( \lambda \to 1 \) for \( k = 0, 1, 2, \ldots \). Moreover, \( |\hat{h}_\lambda(k)| \leq B \) for all \( 0 < \lambda < 1 \) and \( k = 0, 1, 2, \ldots \).

We claim that \( H_\lambda(z) \to 0 \) in \( \mathcal{S}(\alpha) \) as \( \lambda \to 1 \). For

\[
(4.1) \quad \|H_\lambda\|_c = \sum_{k=0}^\infty |\hat{h}_\lambda(k)| e^{-ck} = \sum_{k=0}^N |\hat{h}_\lambda(k)| e^{-ck} + \sum_{N+1}^\infty |\hat{h}_\lambda(k)| e^{-ck}.
\]

Choose \( N \) large enough so that \( \sum_{N+1}^\infty e^{-ck} \leq \epsilon/2B \) and \( \lambda \) so close to 1 that \( |\hat{h}_\lambda(k)| \leq \epsilon/2(N+1) \) for \( k = 0, 1, \ldots, N \). Substitution in (4.1) gives

\[
(4.2) \quad \|H_\lambda\|_c \leq \epsilon \quad \text{for } \lambda \text{ sufficiently close to 1.}
\]

By an argument similar to the one given in Lemma 1, the fact that \( S^{-1}_\phi C_z = \overline{H_\lambda(z)}h_\lambda C_z \) and (4.2) one easily obtains

\[
\lim_{\lambda \to 1} \| S^{-1}_\phi C_z \|_c = \lim_{\lambda \to 1} \| \overline{H_\lambda(z)}h_\lambda C_z \|_c \leq \lim_{\lambda \to 1} \| H_\lambda \|_c = 0
\]

for some \( \delta > 0 \) and a constant \( A \) that depends on \( z \) and \( \delta \) but not on \( \lambda \).

**Lemma 4.** Let \( f \in \mathcal{S}_\phi \) and \( g_\lambda = S^{-1}_\phi (S_\phi f - I)f \), then \( \lim_{\lambda \to 1} g_\lambda = f \) in \( \mathcal{S}_\phi \).

**Proof.** As a consequence of Corollary 3.2 of [5], \( L^\infty(\partial\mathbb{D}) \) is dense in \( \mathcal{S}_\phi \). From this and the fact that \( \{C_z : z \in \mathbb{D}\} \) is a fundamental set in \( L^2(\partial\mathbb{D}) \) and that the embedding \( L^2(\partial\mathbb{D}) \to \mathcal{S}_\phi \) is continuous, we conclude that \( \{C_z : z \in \mathbb{D}\} \) is also a fundamental set in \( \mathcal{S}_\phi \). This fact together with Lemmas 2 and 3 may now be used to show that \( \lim_{\lambda \to 1} \| S^{-1}_\phi f \|_c = 0 \). But \( S^{-1}_\phi f = f - g_\lambda \) which yields the result.

Using arguments parallel to those given in [4 and 1], the reader should be able to finish the proof of the following theorem.

**Theorem.** Let \( E \) be an open arc of \( \partial\mathbb{D} \). Let \( F(z) \in \mathcal{S}(\alpha) \) and \( f \) be its distributional boundary value on \( \partial\mathbb{D} \). Suppose that \( g \) is the restriction of \( f \) to \( E \). For each \( 0 < \lambda < 1 \) define an analytic function \( G_\lambda(z) \) on \( \mathbb{D} \) by

\[
G_\lambda(z) = H_\lambda(z) \left\langle g, (\phi_\lambda - 1)h_\lambda C_z \right\rangle.
\]

Then, \( \lim_{\lambda \to 1} G_\lambda(z) = F(z) \) in \( \mathcal{S}(\alpha) \). In particular, \( \lim_{\lambda \to 1} G(z) = F(z) \) uniformly on compact subsets of \( \mathbb{D} \).

It is well known that functions in \( \mathcal{S}(\alpha) \) can have radial limits equal to zero on a set \( E \) of positive measure (in fact, even on a set of measure \( 2\pi \)) without being
\[ M(r, f) = \sup_{|z|=r} |f(z)| \] and \( \beta = \alpha/(1 - \alpha) \), then any function \( f(z) \in \mathcal{K}(1) = \bigcup_{0<\alpha<1} \mathcal{K}(\alpha) \) can be recovered from the restriction of its distributional boundary value to an open arc \( E \). The technique we use is a modification of the ones given by Patil and Zarantonello.

2. Preliminaries. For \( 0 < \alpha < 1 \), let \( \mathcal{K}(\alpha) \) be the space of all holomorphic functions \( F(z) \) in \( \mathbb{D} \) such that \( F(z) = \sum_{n=0}^{\infty} a_n z^n \) with \( a_n = O(e^{\alpha(n^n)}) \) as \( n \to \infty \). The topology of \( \mathcal{K}(\alpha) \) is the topology induced by the seminorms

\[
\| F \|_c = \sum_{n=0}^{\infty} |a_n| e^{-cn^n} < \infty \quad \text{for } c > 0.
\]

In [5 and 6], we showed the following facts.

1. \( F(z) \in \mathcal{K}(\alpha) \) if and only if

\[
\| F \|_c = \int_{0}^{1} M(r, f) \exp\left( -\frac{c}{(1 - r)^\beta} \right) dr < \infty \quad \text{for all } c > 0.
\]

Moreover, the two families of seminorms \( \| \cdot \|_c \) and \( \| \cdot \|_c^* \) are equivalent.

2. Provided with that topology, \( \mathcal{K}(\alpha) \) becomes a Fréchet-Montel space whose topology is stronger than the topology of uniform convergence on compact subsets of \( \mathbb{D} \).

3. If we denote by \( \mathcal{A}(\mathbb{D}) \) the algebra of functions \( G(z) \) that are analytic in \( \mathbb{D} \) and continuous in \( \mathbb{D} \) such that \( G(z) = \sum_{n=0}^{\infty} b_n z^n \), \( b_n = O(e^{-cn^n}) \) as \( n \to \infty \) for some \( c > 0 \), then for any \( \phi \in \mathcal{K}^*(\alpha) \) (the dual space of \( \mathcal{K}(\alpha) \)), there exists a unique function \( G(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{A}(\mathbb{D}) \) such that

\[
\phi(F) = \lim_{r \to 1} \int_{\partial \mathbb{D}} F(re^{i\theta})G(e^{-i\theta}) \, d\mu = \sum_{n=0}^{\infty} a_n b_n.
\]

Conversely, any \( G(z) \in \mathcal{A}(\mathbb{D}) \) defines a continuous linear functional on \( \mathcal{K}(\alpha) \) via (2.2).

Let \( \mathcal{A}_\alpha = \mathcal{A}(\partial \mathbb{D}) \) be the space of all \( C^\infty \)-functions \( \phi \) on \( \partial \mathbb{D} \) such that

\[
\phi(\theta) = \sum_{n=0}^{\infty} b_n e^{in\theta}
\]

with \( b_n = O(e^{-cn^n}) \) as \( n \to \infty \) for some \( c > 0 \).

Another useful characterization of the class \( \mathcal{A}_\alpha \) is that \( \phi \in \mathcal{A}_\alpha \) if and only if

\[
\sup_{0 < |\theta| < 2\pi} |\phi(n)(\theta)| \leq K n^{\alpha/n^{\alpha}}
\]

for some constants \( K \) and \( R \).

Indeed, the class \( \mathcal{A}_\alpha \) is the same as the class \( C((n!)^{1/\alpha}) \) in the notation of [2].

It follows from the Denjoy-Carleman theorem that \( \mathcal{A}_\alpha \) is a nonquasianalytic class of functions on \( \partial \mathbb{D} \). A topology is defined on \( \mathcal{A}_\alpha \) by means of the fundamental system of neighborhoods of the origin

\[
V(\lambda) = \left\{ \phi \mid \phi(\theta) = \sum_{n=0}^{\infty} b_n e^{in\theta}, b_n = O(e^{-\lambda_n n^n}) \right\}
\]

where \( \lambda = (\lambda_n)_{n=0}^{\infty} \) and \( \lambda_n \downarrow 0 \).

The strong dual \( \mathcal{A}_\alpha^* \) of \( \mathcal{A}_\alpha \) is a space of Beurling distributions which are more general than Schwartz distributions (see [5 and 6] for references). One can verify the following.
(1) Every Beurling distribution $f$ has a Fourier series expansion $f = \sum_{\infty} a_n w^n$ that converges weakly to it, where $w = e^{i\theta}$ and $a_n = \langle f, w^n \rangle$.

(2) A necessary and sufficient condition that $f = \sum_{\infty} a_n w^n \in \mathcal{B}^*$ is that $a_n = O(e^{\alpha|n^\gamma|})$ as $n \to \infty$. Furthermore, if $\psi = \sum_{\infty} b_n w^n \in \mathcal{B}^*$ then $\langle f, \psi \rangle = \sum_{\infty} a_n b_n$.

(3) If $F(z) = \sum_{n=0}^\infty a_n z^n \in \mathcal{K}(a)$, then $f = \sum_{n=0}^\infty a_n w^n$ is a Beurling distribution belonging to the space $\mathcal{B}^*$. In addition to that (cf. [5, Corollary 3.1])

$$F_r(w) = F(rw) \to f \quad \text{in} \quad \mathcal{B}^*.$$

As in [4], we say that $f$ is the distributional boundary value of $F(z)$ and $F(z)$ is the holomorphic extension of $f$. The space of all distributional boundary values of functions in $\mathcal{K}(a)$ will be denoted by $\mathcal{B}^*$. We provide $\mathcal{B}^*$ with a topological structure isometric to that of $\mathcal{K}(a)$ by setting

$$\|f\|_c = \|F\|_c.$$

From (2.1) and (2.2), one can easily see that the topology of $\mathcal{B}^*$ is stronger than the topology it inherits from $\mathcal{B}^*$.

From Cauchy's formula

$$F(rz) = \int_{\partial \Omega} \frac{F(w)}{1 - wz} d\mu(w)$$

and equation (2.3), it follows that $F(z) = \langle f, C_z \rangle$ where $C_z(w) = 1/(1 - wz)$.

### 3. Toeplitz operators on $\mathcal{B}^*$

Since $\mathcal{B}^*$ is an algebra [2], we define the multiplication operator $M_\phi$ for $\phi \in \mathcal{B}^*$ by

$$\langle M_\phi f, \psi \rangle = \langle f, \phi \psi \rangle$$

where $f \in \mathcal{B}^*$ and $\psi \in \mathcal{B}^*$.

The projection operator $P: \mathcal{B}^* \to \mathcal{B}^*$ is formally given by

$$P \left( \sum_{n=0}^\infty a_n w^n \right) = \sum_{n=0}^\infty a_n w^n$$

and hence the Toeplitz operator $S_\phi$ can now be defined on $\mathcal{B}^*$ as $S_\phi = PM_\phi$ for $\phi \in \mathcal{B}^*$.

**Lemma 1.** Let $\phi \in \mathcal{B}^*$ and $f \in \mathcal{B}^*$, then:

(i) For any $c > 0$ there exists $\tilde{c}$ such that

$$\|S_\phi f\|_c \leq K(\phi, c)\|f\|_c$$

where $K(\phi, c)$ is a constant that depends only on $c$ and $\phi$ but not on $f$.

(ii) $S_\phi f$ is the distributional boundary value of the analytic function $\langle M_\phi f, C_z \rangle$.

**Proof.** (i) Let $f = \sum_{n=0}^\infty a_n w^n$ and $\phi = \sum_{n=0}^\infty b_n w^n$, then $S_\phi f = \sum_{k=0}^\infty \sum_{n=0}^\infty a_n b_{k-n} w^k$. Hence,

$$\|S_\phi f\|_c \leq \sum_{k=0}^\infty \sum_{n=0}^\infty |a_n| |b_{k-n}| e^{-c|k|} = \sum_{n=0}^\infty |a_n| \sum_{k=-n}^{\infty} |b_k| e^{-c|k+n|}$$

$$= \sum_{n=0}^\infty |a_n| \sum_{k=-n}^{\infty} |b_k| e^{-c|k+n|} + \sum_{n=0}^\infty |a_n| \sum_{k=0}^{\infty} |b_k| e^{-c|k+n|} = I_1 + I_2.$$
identically zero i.e., the class \( \mathcal{H}(\alpha) \) does not have the uniqueness property. However, the class \( \mathcal{H}(\alpha) \) possesses another uniqueness property which we state as a corollary.

**Corollary.** Let \( F(z) \in \mathcal{H}(\alpha) \) and \( f \) be its distributional boundary value. If \( f = 0 \) (in the sense of distributions) on an open arc \( E \), then \( F(z) \) is identically zero.

**Remarks.** (i) One may ask how far can the result be extended? To answer this question, let us consider the class \( \mathcal{C}(\omega) \) of all analytic functions \( F(z) = \sum_{n=0}^{\infty} a_n z^n \) in \( \mathbb{D} \) with \( a_n = O(e^{\alpha(\omega(n))}) \) as \( n \to \infty \) where \( \omega(x) \) is continuously differentiable and monotonically increasing on \( [0, \infty) \). Certainly, our technique fails if \( \sum_{n=0}^{\infty} \omega(n)/n^2 = \infty \), since in this case the class \( \mathcal{C}(\omega) \) is quasianalytic by the Denjoy-Carleman theorem. Therefore, if the function \( \psi(w) = 0 \) on a set of positive measure, it is identically zero on \( \partial \mathbb{D} \) and hence \( G_{\lambda}(z) \) is identically zero for all \( 0 < \lambda < 1 \).

(ii) The class \( N^+ \) provided with the topology induced by the metric

\[
(f, g) = \int_{\partial \mathbb{D}} \log(1 + |f(e^{i\theta}) - g(e^{i\theta})|) \, d\mu
\]

where \( f \) and \( g \in N^+ \) is a subspace of \( \mathcal{H}(1/2) \).

From the main theorem, it follows that if \( F \in N^+ \), then \( G_{\lambda}(z) \to F(z) \) in \( \mathcal{H}(1/2) \). It would be interesting to know whether \( G_{\lambda}(z) \to F(z) \) in \( N^+ \).

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**References**


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