WHEN IS A KIRILLOV ORBIT A LINEAR VARIETY?

RAINER FELIX

ABSTRACT. It is well known that a Kirillov orbit is a linear variety if and only if the corresponding irreducible representation is square integrable modulo its kernel ([1], Theorem 1.1). Now we give a new representation-theoretic criterion for a Kirillov orbit being a linear variety in terms of weak containment and tensor products of group representations.

Let $G$ be a nilpotent simply connected Lie group and $\mathfrak{g}$ its Lie algebra. For $\pi \in \hat{G}$, the set of equivalence classes of irreducible unitary representations of $G$, let $\Omega_\pi$ be the corresponding Kirillov orbit in the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$. The orbit corresponding to the one-dimensional identity representation $1$ of $G$ is $\{0\}$, and the orbit corresponding to the conjugate representation $\overline{\pi}$ of $\pi$ is $-\Omega_\pi$. For $\pi, \rho \in \hat{G}$ an element $\sigma \in \hat{G}$ is weakly contained in the tensor product $\pi \otimes \rho$ of $\pi$ and $\rho$ if and only if $\Omega_\sigma \subseteq \Omega_\pi + \Omega_\rho$ [2, Lemma 2.1]. Therefore $1$ is weakly contained in $\pi \otimes \overline{\pi}$ for every $\pi$ (compare [4, Lemma 1]). Now we investigate under which condition $1$ is weakly contained in $\pi \otimes \rho$ for $\rho \neq \pi$ (compare [3, 2]).

**Theorem.** The following properties of $\pi \in \hat{G}$ are equivalent:

(i) $\Omega_\pi$ is a linear variety;

(ii) if $1$ is weakly contained in $\pi \otimes \rho$ for $\rho \in \hat{G}$, then $\rho = \pi$.

**Proof.** Let $\Omega_\pi$ be a linear variety and $0 \in \Omega_\pi - \Omega_\rho$. Then $\Omega_\pi = p + M$, where $p \in \mathfrak{g}^*$ and $M$ is a linear subspace of $\mathfrak{g}^*$, which is invariant under the coadjoint representation. The quotient representation of the coadjoint representation in the quotient space $\mathfrak{g}^*/M$ is unipotent, consequently the orbits in $\mathfrak{g}^*/M$ are closed. Thus $k(\Omega_\rho) \subseteq \mathfrak{g}^*/M$ is closed, where $k : \mathfrak{g}^* \to \mathfrak{g}^*/M$ is the canonical mapping. By $0 \in \Omega_\pi - \Omega_\rho$ we follow $k(p) \in k(\Omega_\rho)$, then $k(p) = k(q)$ with $q \in \Omega_\rho$, then $q \in p + M = \Omega_\pi$.

Conversely, assuming that $\Omega_\pi$ is not a linear variety, we have to show that there exists an orbit $\Omega \neq \Omega_\pi$ with $0 \in \overline{\Omega_\pi} - \overline{\Omega}$. By assumption, there are points $p, q \in \Omega_\pi$ such that the line segment $[p, q]$ joining $p$ and $q$ is not contained in $\Omega_\pi$. Let $\text{Exp} \mathbb{R} X, X \in L(\mathfrak{g}^*)$, be a one-parameter subgroup of the coadjoint group of $G$ with $(\text{Exp} X)p = q$. Because $X$ is a nilpotent endomorphism, there is a greatest natural
number \( n \) such that \( X^n p \neq 0 \). Let \( m \leq n \) be the smallest number such that \( \text{Exp} \mathbb{R} X p + F \subseteq \Omega_* \), \( F \) being the linear hull of the vectors \( X^{m+1} p, X^{m+2} p, \ldots, X^n p \) in \( \mathbb{O}^* \).

(The linear subspace \( F \) of \( \mathbb{O}^* \) can be regarded as the "flat part" of \( \Omega_* \) along \( \text{Exp} \mathbb{R} X p \).) In view of \([ p, q ] \not\subseteq \Omega_* \) we conclude \( m > 1 \). Thus we can find \( \alpha, \beta \in \mathbb{R} \) and \( f \in F \) such that the point \( q := (\text{Exp} \alpha X)p + \beta X^n p + f \) does not belong to \( \Omega_* \).

Let \( \Omega \) be the coadjoint orbit of \( q \). The subspace \( F \) of \( \mathbb{O}^* \) is invariant under \( X \). Therefore we can form the endomorphism \( X_F \), associated with \( X \), of the quotient space \( \mathbb{O}^*/F \). In order to see that \( 0 \in \Omega_* - \Omega \) it is enough to show that

\[
0 \in k(\text{Exp} \mathbb{R} X p) - k(\text{Exp} \mathbb{R} X q) = \text{Exp} \mathbb{R} X_F k(p) - \text{Exp} \mathbb{R} X_F k(q)
\]

in \( \mathbb{O}^*/F \), \( k : \mathbb{O}^* \rightarrow \mathbb{O}^*/F \) being the canonical mapping. Let us define the sequence

\[
x_r := (\text{Exp} \gamma_r X_F)k(p) \in \text{Exp} \mathbb{R} X_F k(p)
\]

with \( \gamma_r := (\nu^m + m! \beta)^{1/m} \) and the sequence

\[
x'_r := (\text{Exp}(\nu - \alpha) X_F)k(q) \in \text{Exp} \mathbb{R} X_F k(q).
\]

Then

\[
x_r - x'_r = \sum_{r=0}^{m} \gamma_r^r - \nu^r \frac{X_F^r k(p)}{r!} - \beta X_F^r k(p)
\]

is a null sequence, because \( \gamma_r^r - \nu^r \) is a null sequence in \( \mathbb{R} \) for \( r < m \).

**Corollary.** \( \pi \in \hat{G} \) is square integrable modulo its kernel if and only if any element \( \rho \in \hat{G} \), for which 1 is weakly contained in \( \pi \otimes \rho \), must be equal to \( \pi \).

Maybe the assertion of this corollary remains true for more general locally compact groups.

**Acknowledgements.** The author is indebted to E. Kaniuth for the conjecture that all orbits are linear varieties iff for all \( \pi, \rho \in \hat{G} \) the trivial representation 1 is weakly contained in \( \pi \otimes \rho \) only for \( \pi = \rho \) (compare [4, Examples]).

**References**


**Institut für Mathematik der Technischen Universität, D-8 München 2, Arcisstrasse 21, Postfach 20 24 20, Germany**