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A PATHOLOGICAL AREA PRESERVING C^∞ DIFFEOMORPHISM OF THE PLANE

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ABSTRACT. The pseudocircle P is an hereditarily indecomposable planar continuum. In particular, it is connected but nowhere locally connected. We construct a C^∞ area preserving diffeomorphism of the plane with P as a minimal set. The diffeomorphism f is constructed as an explicit limit of diffeomorphisms conjugate to rotations about the origin. There is a well-defined irrational rotation number for $f|P$ even though $f|P$ is not even semi-conjugate to a rotation of S^1. If we remove the requirement that our diffeomorphisms be area preserving, we may alter the example so that P is an attracting minimal set.

The complexity of a dynamical system is reflected in part by its invariant sets. We consider here a simple dynamical system, the action of a diffeomorphism on $\mathbf{R}^2$. Pathology abounds in the compact connected subsets of $\mathbf{R}^2$, and we show that this pathology will occur in the minimal sets of diffeomorphisms, even if we restrict ourselves to those which are C^∞ and area preserving.

We choose the pseudocircle P (defined below) as our model of extreme pathology. Its key feature is that it is hereditarily indecomposable. Indecomposable means that P cannot be written as the union of two proper compact connected subsets, and hereditarily indecomposable means that every compact connected subset of P is indecomposable. (P is, for instance, nowhere locally connected.) What makes P tractable in spite of this behavior, is that Bing's construction of pseudocircles **[B2]** is both simple and malleable. It is an infinite construction allowing choices at each stage. In light of **[F]**, the resulting space is, to a great extent, independent of the choices.

Using an infinite limit, we construct an embedding of P in $\mathbf{R}^2$ and a C^∞ area preserving diffeomorphism $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ with P as a minimal set. (One may take the domain of f to be an annulus A^2.) There are two other features of f that are worth mentioning. P is defined as the intersection of annuli $P = \bigcap_{n=1}^{\infty} A_n$ where each inclusion $A_{n+1} \to A_n$ is a homotopy equivalence. It therefore makes sense to speak of a rotation number for f and indeed f has a well-defined irrational rotation number. Nonetheless, f is not semiconjugate to a homeomorphism of S^1. Second, if one is willing to consider C^∞ diffeomorphisms which do not preserve area, f is easily perturbed to $f'\colon \mathbf{R}^2 \to \mathbf{R}^2$ with P as an attracting minimal set.

It is especially relevant that f is area preserving, as there is a long history of interest in area preserving diffeomorphisms of a surface. In particular, Birkhoff **[Bir]** (see also the recent work of Mather **[M]**) studied invariant sets which were the frontiers of invariant, open, simply connected regions, and gave criterion forcing

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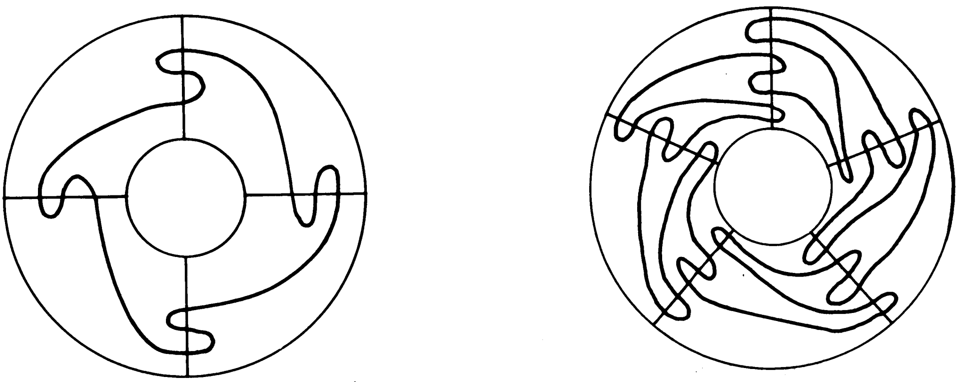
the invariant set to be a Jordan curve. To my knowledge, ours is the first such example where the area preserving diffeomorphism f acts transitively on the invariant set P but P is not a Jordan curve.

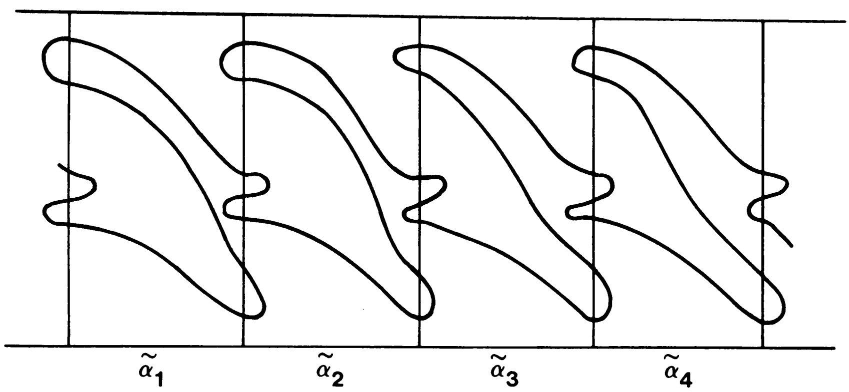
I would like to thank W. Thurston for pointing out that f is not semiconjugate to a homeomorphism of S^1, and J. Mather for several interesting conversations.

The construction. Begin with C_1, the unit circle in $\mathbf{R}^2$, and $\epsilon_1 > 0$. Divide C_1 into p_1 equal pieces $D_1, \ldots, D_{p_1}$, each of diameter less than ϵ_1, and thicken radially to $A_1 \cong D_1 \times [-1,1]$ so that each $a_i \cong D_i \times [-1,1]$ still has diameter less than ϵ_1. Define $f_1 \colon \mathbf{R}^2 \to \mathbf{R}^2$ so that $f_1 | C_1 \times [-1,1] = R_{\alpha_1} \times$ identity where $\alpha_1 = 1/p_1$ and R_α is rotation through the angle α.



$p_1=4$ $C_2 \subset A_1$ $p_1=5$

FIGURE 1



$\tilde{C}_2 \subset \tilde{A}_1$ $p_1=5$

FIGURE 2

Embed a circle C_2 in A_1 which is crooked with respect to $a_1, \ldots, a_{p_1}$ and invariant with respect to f_1. A more general definition of crookedness is given in [**B1**] but for our purposes, the following suffices. The inclusion $C_2 \to A_1$ is a homotopy equivalence. Consider the universal covers $\tilde{C}_2 \subset \tilde{A}_1$. The division of A_1 into a_i, $i = 1, \ldots, p_1$, lifts to a division of $\tilde{A}_1$ into $\tilde{a}_i$, $i = -\infty, \ldots, \infty$. $\tilde{C}_2$ is crookedly embedded in A_1 with respect to $a_1, \ldots, a_{p_1}$ if any segment α of $\tilde{C}_2$ running from $\tilde{a}_i$ to $\tilde{a}_j$ $(p_1 > j - i > 2)$ can be written as a composition $\alpha = \alpha_1 \cdot \alpha_2 \cdot \alpha_3$ where

α_1 runs from $\tilde{a}_i$ to $\tilde{a}_{j-1}$, α_2 runs from $\tilde{a}_{j-1}$ to $\tilde{a}_{i+1}$, and α_3 runs from $\tilde{a}_{i+1}$ to $\tilde{a}_j$.

For our construction, C_2 must be invariant by f_1. Let $T \cong c \times [-1,1] \subset C_1 \times [-1,1]$ for some $c \in C_1$, and let $T \cdot S$ denote the algebraic intersection of T with an arc S. Choose a crooked arc S_1 (see Figure 3) which spirals down toward the inner boundary component of A_1 and which satisfies $T \cdot S_1 = p_1 - 2$. (An arc S in $C_1 \times [-1,1]$ is crooked if any segment α with $T \cdot \alpha = k > 2$ can be decomposed into 3 subarcs $\alpha = \alpha_1 \cdot \alpha_2 \cdot \alpha_3$ such that $T \cdot \alpha_1 = T \cdot \alpha_3 = -T \cdot \alpha_2 + 1 = k - 1$.) Push all but the first loop of S_1 slightly above itself obtaining a crooked arc S_2 spiraling toward the outer boundary of A_1 and satisfying $T \cdot S_2 = p_1 - 3$. Connect the endpoints of S_1 to those of S_2 without intersecting T. This gives a degree one circle C_2' in A_1 whose p_1-fold cover C_2 is both crooked with respect to $a_1, \ldots, a_{p_1}$ and invariant by $f_1 = R_{\alpha_1} \times$ identity.

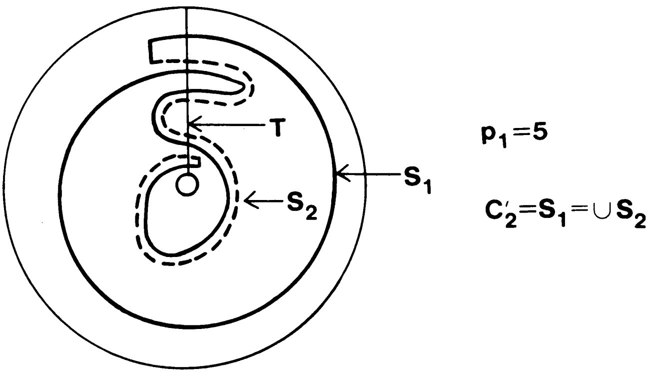


FIGURE 3

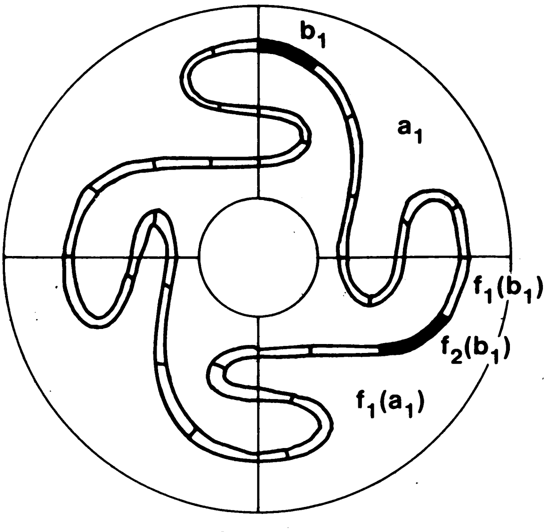
We return to the construction of f and P. Choose $\epsilon_2 > 0$ and a rational number $\alpha_2 = \alpha_1 + 1/m_2$ where m_2 is a large positive integer (conditions on how large m_2 must be, will be enumerated later). Since C_2 is smoothly embedded in $C_1 \times [-1,1]$ and $f_1 | C_1 \times [-1,1] = R_{\alpha_1} \times \text{id}$, C_2 can be thickened to an annulus $A_2 \cong C_2 \times [-1/2, 1/2] \subset C_2 \times [-1,1] \subset C_1 \times [-1,1]$ (this last inclusion does not map fibers into fibers) so that $f_1 | C_2 \times [-1,1] = R_{\alpha_1} \times \text{id}$ and so that $C_2 \times [-1,1]$, with the metric induced from $C_1 \times [-1,1]$, is isometric to $S_l^1 \times [-\epsilon_2'/4, \epsilon_2'/4]$, where S_l^1 is the circle of length l ($=$ length of C_2) in $\mathbf{R}^2$ and $\epsilon_2' < \epsilon_2$.

Define $f_2 \colon \mathbf{R}^2 \to \mathbf{R}^2$ by

$$f_2 = \begin{cases} f_1 \text{ on } \mathbf{R}^2 - (C_2 \times [-1,1]), \\ R_{\alpha_2} \times \text{id on } C_2 \times [-1/2, 1/2], \\ R_{\Phi(|t|)} \times \text{id on } C_2 \times ([-1,-1/2] \cup [1/2,1]) \\ \text{where } \Phi\colon [1/2,1] \to [\alpha_2, \alpha_1] \text{ is smooth and is} \\ \text{constant on neighborhoods of } \{1/2\} \text{ and } \{1\}. \end{cases}$$

Note that f_2 is area preserving and that if m_1 is sufficiently large, then f_2 is ϵ_2-close to f_1 in the C^2 topology. Let p_2 be the period of R_{α_2} and partition A_2 into p_2 pieces $b_1, \ldots, b_{p_2}$ transitively permuted by f_2.

This completes a cycle in the inductive construction of f_n and A_n. To construct f_3 and A_3, repeat the steps above: embed C_3 crookedly in A_2 with respect to



$A_1 \subset A_2$

FIGURE 4

$b_1, \ldots, b_{p_2}$, and invariantly with respect to f_2; thicken C_3 to A_3 and let $f_3|A_3$ be f_2 followed by a slight rotation of A_3 in its own annular structure, not that of A_2; partition A_3 with respect to f_3.

Iterate this to construct f_n, A_n and ϵ_n with the following additional properties:

(1) $f_n\colon \mathbf{R}^2 \to \mathbf{R}^2$ is an area preserving C^∞ diffeomorphism which is ϵ_n-close to f_{n-1} in the C^n topology.

(2) A_n is partitioned into p_n elements of diameter less than ϵ_n. These elements are transitively permuted by f_n.

(3) The ϵ_n's are so small that $f = \lim_{n\to\infty} f_n$ is a C^∞ area preserving diffeomorphism and $|f^i(x) - f_n^i(x)| < \epsilon_n$ for $0 \leq i \leq p_n$ and all $x \in A_n$.

Conditions (2) and (3) guarantee that f acts minimally on $P = \bigcap_{n=1}^\infty A_n$ which, by [F], is the pseudocircle.

It remains to verify the two additional features of f mentioned in the introduction. It is easy to construct a C^∞ diffeomorphism $h\colon S^1 \times [0,1) \to A_1 - P$ such that each $h(S^1 \times \{t\})$ is invariant by f. Perturbing f slightly in this induced structure we easily construct a C^∞ diffeomorphism $f'\colon \mathbf{R}^2 \to \mathbf{R}^2$ with P as an attracting minimal set.

Showing that $f|P$ is not semiconjugate to a homeomorphism of S^1 is more involved and we will only outline the argument. Suppose then that there exists a homeomorphism $g\colon S^1 \to S^1$ and an onto map $\Pi\colon P \to S^1$ such that

$$\begin{array}{ccc} P & \xrightarrow{f} & P \\ \Pi\downarrow & & \downarrow\Pi \\ S^1 & \xrightarrow{g} & S^1 \end{array}$$

commutes.

Let $\gamma_n\colon A_n \to S^1$ be the projection of the annulus to its core circle and let $\tilde{\gamma}_n\colon \tilde{A}_n \to \mathbf{R}$ be the lift of γ_n to the corresponding universal covers. The inclusions $A_{n+1} \subset A_n$ are homotopy equivalences, so $\tilde{A}_{n+1} \subset \tilde{A}_n$ and $\tilde{P} = \bigcap_{n=1}^\infty \tilde{A}_n$ is an infinite cyclic cover of P. The commutative diagram above lifts to another commutative diagram.

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\tilde{f}} & \tilde{P} \\ \tilde{\Pi}\downarrow & & \downarrow\tilde{\Pi} \\ \mathbf{R} & \xrightarrow{\tilde{g}} & \mathbf{R} \end{array}$$

It follows that $\tilde{\Pi}$ is proper (if the image of $\tilde{\Pi}$ were compact, then g would have a fixed point and Π would not map P onto S^1) and hence that: $\forall n > 0\, \exists B_n$ such that $|\tilde{\gamma}_n(\tilde{x}) - \tilde{\gamma}_n(\tilde{y})| < 1 \Rightarrow |\tilde{\gamma}_n\tilde{f}^k(\tilde{x}) - \tilde{\gamma}_n\tilde{f}^k(y)| < B_n$ for $\forall k$ and $\forall \tilde{x}, \tilde{y} \in \tilde{P}$.

In contrast to this, consider the way that a fundamental domain of $\tilde{A}_m$ is spreading out in $\tilde{A}_n$ for $m > n$. It is easy to check that if the partitioning of A_k is sufficiently fine for all k, then a fundamental domain of $\tilde{A}_m$ will intersect $m - n + 1$ fundamental domains of $\tilde{A}_n$. In Figure 5 the crookedness of $\tilde{A}_m$ has been surpressed.

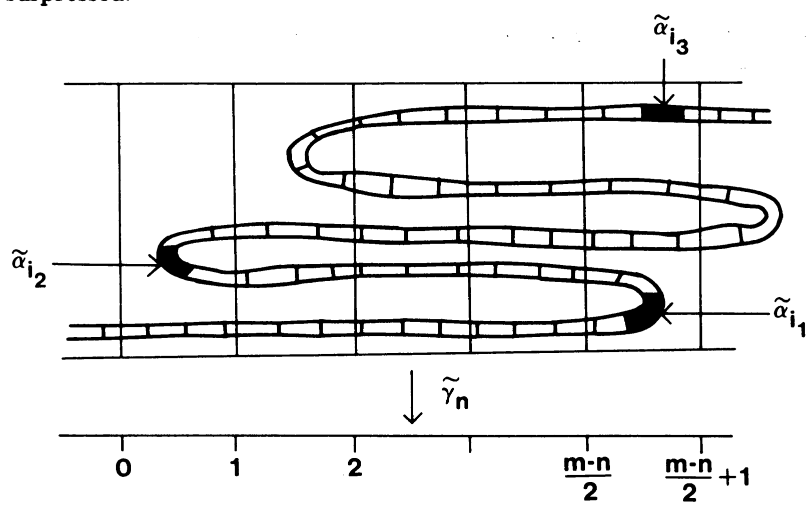


FIGURE 5

Let $\tilde{a}_i$, $i = -\infty, \ldots, \infty$, be the partition of $\tilde{A}_m$. Choose $i_1 < i_2 < i_3$ so that

(i) $\tilde{\gamma}_n(\tilde{a}_{i_1}), \tilde{\gamma}_n(\tilde{a}_{i_3}) \subset [(m-n)/2, (m-n+1)/2]$,

(ii) $\tilde{\gamma}_n(\tilde{a}_{i_2}) \subset [0, 1)$,

(iii) $\tilde{\gamma}_n(\tilde{a}_i) > (m-n)/2$ for $i > i_3$.

Since f_m permutes the p_m elements of the partition of A_m, there exist $1 \leq k \leq p_m$ and $j \in \mathbf{Z}$ such that $t^j \cdot f_m^k(\tilde{a}_{i_1}) = \tilde{a}_{i_2}$ where t generates the deck transformations of $\tilde{A}_m$. Thus for any $\tilde{x} \in \tilde{a}_{i_1}$ and $\tilde{y} \in \tilde{a}_{i_3}$, $|\tilde{\gamma}_n(\tilde{x}) - \tilde{\gamma}_n(\tilde{y})| < 1$ and $|\tilde{\gamma}_n\tilde{f}_m^k(\tilde{x}) - \tilde{\gamma}_n\tilde{f}_m^k(\tilde{y})| > (m-n)/2 - 1$. If the ϵ_i's are sufficiently small, then the last equation will hold with $\tilde{f}$ replacing $\tilde{f}_m$. This contradiction completes the proof that $f\colon P \to P$ is not semiconjugate to a homeomorphism of S^1.

Finally, we indicate why $f|P$ has a well-defined rotation number. It follows from the construction of P that $|\tilde{\gamma}_n - \tilde{\gamma}_1| < 2n$. to show that $\alpha = \lim_{n\to\infty} \alpha_n$ is the rotation number of $f|P$, we must show that

$$\lim_{k\to\infty} \frac{\tilde{\gamma}_1\tilde{f}^k(\tilde{x}) - \tilde{\gamma}_1(\tilde{x})}{k} = \alpha \quad \text{for all } \tilde{x} \in \tilde{P}.$$

Fix $k > 0$ and choose n so that $2^{n-1} \leq k < 2^n$. By definition, $\tilde{\gamma}_n \tilde{f}_n^k(\tilde{x}) - \tilde{\gamma}_n(\tilde{x}) = k\alpha_n$. If the ϵ_i's are sufficiently small, then $|\tilde{\gamma}_1 \tilde{f}^k - \tilde{\gamma}_1 \tilde{f}_n^k| < 1$ for $0 \leq k \leq 2^n$. Thus

$$\left| \frac{\tilde{\gamma}_1 \tilde{f}^k(\tilde{x}) - \tilde{\gamma}_1(\tilde{x})}{k} - \alpha \right| \leq |\alpha - \alpha_n| + \frac{4_n}{k} + \frac{1}{k},$$

which completes the proof.

REFERENCES

[B1] R. H. Bing, *A homogeneous indecomposable plane continuum*, Duke Math. J. **15** (1948), 729–742.

[B2] ——, *Concerning hereditarily indecomposable continua*, Pacific J. Math. **1** (1951), 43–51.

[Bir] G. D. Birkhoff, *Surface transformations and their dynamical applications*, Acta Math. **43** (1922), 1–119.

[F] L. Fearnley, *The pseudo-circle is unique*, Trans. Amer. Math. Soc. **149** (1970), 45–64.

[M] J. N. Mather, *Invariant subsets for area preserving homeomorphisms of surfaces*, preprint.

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