PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 1, September 1982

SHORTER NOTES

The purpose of this department is to publish very short papers of unusually elegant and polished character, for which there is no other outlet.

GAUSS-BONNET THEOREMS FOR NONCOMPACT SURFACES

STEVEN ROSENBERG

The aim of this note is to give short proofs of the following two theorems, due to Cohn-Vossen [3] and Huber [4] respectively.

THEOREM A (GAUSS-BONNET INEQUALITY). *Let M be a finitely connected complete noncompact Riemannian surface with Gaussian curvature K and area element dA. If $\int_M K\,dA$ is absolutely integrable, then $\chi(M) \geqslant \int_M K\,dA$.*

THEOREM B. *Let M be a finitely connected complete, finite volume noncompact Riemannian surface with $\int_M K\,dA$ absolutely integrable. Then*

$$\chi(M) = \int_M K\,dA.$$

For Thec rem A, see also [1].

Such an M is homeomorphic to a compact surface with p points deleted. A neighborhood of each point is homeomorphic to $S^1 \times \mathbf{R}^+$, and by forming the gradient flow associated to a Morse function on M [5], the metric on the cusp $S^1 \times \mathbf{R}^+$ can be chosen to be of the form $g_{11}(\theta, t)d\theta^2 + g_{22}(\theta, t)dt^2$. Reparametrizing $\mathbf{R}^+$ by arclength puts the metric in the form $g_{11}(\theta, t)d\theta^2 + dt^2$. Since M is complete, the new parameterization ranges over all of $\mathbf{R}^+$.

Let $M_h = M - \cup_1^p\{S^1 \times (h, \infty)\}$, so M_h is just M truncated at height h up each cusp. Then the Gauss-Bonnet Theorem for surfaces with boundary gives $\chi(M_h) = \int_M K\,dA + \int_{\partial M_h} \omega_{12}$, where ω_{12} is the connection one-form associated to an orthonormal frame on M [2]. Since $\chi(M) = \chi(M_h)$, we must show $\lim_{h\to\infty} \int_{\partial M_h} \omega_{12} \geqslant 0$ for Theorem A and $\lim_{h\to\infty} \int_{\partial M_h} \omega_{12} = 0$ for Theorem B.

Received by the editors January 14, 1982.
1980 *Mathematics Subject Classification*. Primary 53C45.

0002-9939/82/0000-0135/$01.25

Picking the orthonormal frame $\theta^1 = \sqrt{g_{11}}\, d\theta$ and $\theta^2 = dt$ gives $\omega_{12} = (d/dt)(\sqrt{g_{11}})d\theta$ via the first structure equation $d\theta^1 = \omega_{12} \wedge \theta^2$. The second structure equation gives $KdA = \Omega_{12} = d\omega_{12} = (d^2/dt^2)(\sqrt{g_{11}})d\theta\, dt$. Since $\int_M K\, dA < \infty$, $\lim_{h\to\infty} \int_{\partial M_h}(d^2/dt)\sqrt{g_{11}}\, d\theta = 0$ or $\lim_{h\to\infty} \int_{\partial M_h}(d/dt)\sqrt{g_{11}}\, d\theta$ is a constant C.

For Theorem B, $\int\int_M \sqrt{g_{11}}\, d\theta\, dt < \infty$ implies $\lim_{h\to\infty} \int_{\partial M_h} \sqrt{g_{11}}\, d\theta = 0$. Now $\lim_{h\to\infty} \int_{\partial M_h} \omega_{12} = \lim_{h\to\infty}(d/dt) \int \sqrt{g_{11}}\, d\theta = C$ forces $C = 0$.

For Theorem A, we need to show $C \geqslant 0$. Since $\int_{\partial M_h} \sqrt{g_{11}}\, d\theta \sim C \cdot h + D$ as $h \to \infty$, if $C < 0$ we get $\int_{\partial M_h} \sqrt{g_{11}}\, d\theta < 0$ for each $h \gg 0$. Since the integrand is positive, this is impossible.

References

1. D. Bleecker, *The Gauss-Bonnet inequality and almost geodesic loops*, Adv. in Math. **14** (1974), 183–194.

2. S.-S. Chern, *On the curvatura integra in a Riemannian manifold*, Ann. of Math. **46** (1945), 674–684.

3. S. Cohn-Vossen, *Kürzeste Wege und totalkrümmung auf Flächen*, Compositio Math. **2** (1935), 69–133.

4. Alfred Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **32** (1957), 13–72.

5. J. Milnor, *Morse theory*, Ann. of Math. Studies, no. 51, Princeton Univ. Press, Princeton, N. J., 1963, p. 12.

Department of Mathematics, Brandeis University, Waltham, Massachusetts 02154