EXTREME POINTS AND $l_1(\Gamma)$-SPACES

NINA M. ROY

ABSTRACT. Let $X$ be a nontrivial real Banach space and let $E_X$ denote the set of extreme points of the closed unit ball $B(X)$.

Theorem 1. $X$ is an $l_1(\Gamma)$-space if and only if (i) span($e$) is an $L$-summand in $X$ for every $e \in E_X$ and (ii) $B(X)$ is the norm closed convex hull of $E_X$.

Theorem 2. Let $X = Y^*$. If (i) span($e$) is an $L$-summand in $X$ for every $e \in E_X$ and (ii) $\{e \in E_X: e(y) = 1\}$ is countable for each $y$ in $Y$ with $\|y\| = 1$, then $X$ is an $l_1(\Gamma)$-space.

By definition, an $L$-projection on a Banach space $X$ is a projection $P$ such that $\|x\| = \|Px\| + \|x - Px\|$ for every $x$ in $X$; the range of $P$ is called an $L$-summand in $X$. An $l_1(\Gamma)$-space is a Banach space which is linearly isometric to the space $l_1(\Gamma)$ of all real-valued summable functions on some set $\Gamma$. Let $X$ be a nontrivial real Banach space and let $E_X$ denote the set of extreme points of the closed unit ball $B(X)$. In this paper we prove (Theorem 1) that $X$ is an $l_1(\Gamma)$-space if and only if (i) span($e$) is an $L$-summand in $X$ for every $e \in E_X$ and (ii) $B(X)$ is the norm closed convex hull of $E_X$. As a consequence we have (Theorem 2) that a dual space $X = Y^*$ is an $l_1(\Gamma)$-space if (i) span($e$) is an $L$-summand in $X$ for every $e \in E_X$ and (ii) $\{e \in E_X: e(y) = 1\}$ is countable for each $y$ in $Y$ with $\|y\| = 1$. The proof of Theorem 2 uses the Bishop-Phelps theorem and a result of J. Bourgain to show that $B(X)$ is the norm closed convex hull of $E_X$. Our paper concludes with an example of a nonseparable space $Y$ which satisfies the hypotheses of Theorem 2 and contains uncountably many $y$ such that $\|y\| = 1$ and $\{e \in E_Y: e(y) = 1\}$ is countably infinite.

In what follows, if $S$ is a subset of a Banach space, then the convex hull of $S$ is denoted by $co S$ and the linear span of $S$ by span $S$. The norm closure of $S$ is denoted by norm-$cl(S)$. All Banach spaces are assumed to be nontrivial.

In Lemmas 1 and 2, $A$ is a real Banach space for which $E_A \neq \emptyset$.

Lemma 1. Let $A$ be a nonempty finite subset of $E_A$ such that span($e$) is an $L$-summand in $X$ for every $e$ in $A$, and let $N = $ span $A$. Then $B(N) = co(A \cup -A)$.

Proof. Since $N = \sum$ span($e$) ($e \in A$), we have that $N$ is an $L$-summand in $X$ and $E_N = A \cup -A$ [I, Propositions 1.13 and 1.15]. Then $B(N) = co(A \cup -A)$ because $A$ is finite.

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LEMMA 2. Assume that span(e) is an L-summand in X for every e in $E_X$. Let $\{e_n: n = 1, 2, \ldots\}$ be a linearly independent subset of $E_X$ and let $x_n \in$ span($e_n$) for $n = 1, 2, \ldots$. If $\sum \|x_n\| < \infty$, then $\sum x_n$ converges and $\|\sum x_n\| = \sum \|x_n\|.

PROOF. The proof follows from the fact that $\|\sum_{k=1}^n x_n\| = \sum_{k=1}^n \|x_n\|$ for all $k$. To obtain the induction step, observe that if $P$ is the L-projection of $X$ onto $N_k = \sum_{n=1}^k$ span($e_n$), then $Pe_{k+1} = 0$ because $e_{k+1} \notin N_k$ by Lemma 1. (An L-projection maps an extreme point to itself or 0.)

THEOREM 1. A real Banach space $X$ is an $l_1(\Gamma)$-space if and only if (i) span(e) is an L-summand in $X$ for every e in $E_X$ and (ii) $B(X)$ is the norm closed convex hull of $E_X$.

PROOF. Suppose that $X$ is an $l_1(\Gamma)$-space. We may assume that $X = l_1(\Gamma)$, where $\Gamma$ is a nonempty set. For each $\gamma$ in $\Gamma$ let $\delta_\gamma$ be the characteristic function of {0}. Then $E_X = \{\pm \delta_\gamma: \gamma \in \Gamma\}$. For each $\gamma$ in $\Gamma$, the map $x \mapsto x\delta_\gamma$ is an L-projection of $X$ onto span($\delta_\gamma$). Thus condition (i) holds (as it does in any $L_1$-space). To prove (ii), let $x \in X$ with $\|x\| \leq 1$. Then there is a countable set $\{\gamma_n\} \subseteq \Gamma$ such that $x(\gamma) = 0$ for $\gamma \notin \{\gamma_n\}$ and $\sum_{n=1}^\infty |x(\gamma_n)| < \infty$. Then $x = \sum_{n=1}^\infty x(\gamma_n)\delta_{\gamma_n}$. For each $k$ let $x_k = \sum_{n=1}^k x(\gamma_n)\delta_{\gamma_n}$. Then $\|x_k\| \leq 1$ and hence by Lemma 1, $x_k \in \text{co}(E_X)$. Therefore $x \in \text{norm-cl}(\text{co}(E_X))$.

For the converse, assume that (i) and (ii) are true. Let $\Gamma$ be a maximal linearly independent subset of $E_X$. Then $E_X = \Gamma \cup -\Gamma$. To see this, suppose there is $e \in E_X$ with $e \notin \Gamma \cup -\Gamma$. Then $e$ is a linear combination of the elements of a finite subset $A$ of $\Gamma$. By Lemma 1, $e \in \text{co}(A \cup -A)$. Then $e \in A \cup -A$ since $e \in E_X$, and we have a contradiction. If $\Gamma = \{e_\gamma\}$, define an operator $T: l_1(\Gamma) \to X$ by $T(f) = \sum f(\gamma) e_\gamma$. By Lemma 2, $T$ is an isometry. Hence its range is closed. By (ii) and the fact that $E_X = \Gamma \cup -\Gamma$, the range of $T$ is dense in $X$. Thus $T$ is surjective.

THEOREM 2. Let $Y$ be a real Banach space such that (i) span(e) is an L-summand in $Y^*$ for every e in $E_{Y^*}$, and (ii) $\{e \in E_{Y^*}: e(y) = 1\}$ is countable for each $y$ in $Y$ with $\|y\| = 1$.

Then $Y^*$ is an $l_1(\Gamma)$-space.

PROOF. By Theorem 1 it suffices to show that $B(Y^*)$ is the norm closed convex hull of $E_{Y^*}$. Let $f \in B(Y^*)$ with $f \neq 0$. By the Bishop-Phelps theorem [2], the set of those $g$ in $Y^*$ which attain their norm is dense in $Y^*$. Hence given $\epsilon > 0$, there is $g$ in $Y^*$ such that $\|f/g\| - g/\|g\| < \epsilon$ and $\|g\| = g(y)$, where $y \in Y$ with $\|y\| = 1$. Let $F_y = \{h \in B(Y^*): h(y) = 1\}$. Then $F_y$ is a weak* compact convex set and $g/\|g\| \in F_y$. Let $E_y$ denote the set of extreme points of $F_y$. Then $E_y \subseteq E_{Y^*}$ because $F_y$ is an extremal subset of $B(Y^*)$. Thus $E_y = \{e \in E_{Y^*}: e(y) = 1\}$. Then $F_y = \text{norm-cl}(\text{co}(E_y))$ because $E_y$ is countable [3]. Let $h \in \text{co}(E_y)$ with $\|h - g/\|g\|\| < \epsilon$. Then $\|h - f/f\| < 2\epsilon$, hence

$$\|f/h - f\| < 2\epsilon \|f\|.$$  

Since $\|f/h \in \text{co}(E_y \cup -E_y)$, it follows that $f \in \text{norm-cl}(\text{co}(E_{Y^*}))$.

We now give an example of a space $Y$ which satisfies the hypotheses of Theorem 2 and contains uncountably many $y$ such that $\|y\| = 1$ and $\{e \in E_{Y^*}: e(y) = 1\}$ is countably infinite.
Let $T$ denote the set of all ordinals less than or equal to the first uncountable ordinal $\Omega$, and let $T$ have the order topology. Let $Y = \{ f \in C(T) : f(\Omega) = 0 \}$. Then $Y^*$ is an $L$-space because $Y$ is an $M$-space; hence the first hypothesis of Theorem 2 is satisfied. For each $t$ in $T$, let the evaluation functional $e_t$ be defined on $Y$ by $e_t(f) = f(t)$ for all $f$ in $Y$. Then $E_{Y^*} = \{ \pm e_t : t \in T, t \neq \Omega \}$. Since each function in $C(T)$ is eventually constant, the second hypothesis of Theorem 2 is satisfied. For each $t$ in $T$ such that $\omega \leq t < \Omega$, let $f_t$ be the characteristic function of the interval $[0, t]$. Then $f_t \in Y$, $\| f_t \| = 1$, and $\{ e \in E_{Y^*} : e(f_t) = 1 \}$ is countably infinite. Clearly the set of functions $f_t$ is uncountable.

In conclusion, we remark that $C(T)^* = l_1(T)$ [4, p. 175], hence the converse of Theorem 2 is false.

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DEPARTMENT OF MATHEMATICS, ROSEMONT COLLEGE, ROSEMONT, PENNSYLVANIA 19010