ABSTRACT. A sequence of Fourier coefficients \( \{ \hat{f}(n) \} \) of a complex function in \( L^1(T) \) is said to be complex quasimonotone if there exists \( \theta_0 \) such that

\[
\Delta \hat{f}(n) + \frac{\alpha}{n} \hat{f}(n) \in \left\{ z : |\arg z| < \theta_0 < \frac{\pi}{2} \right\}
\]

for some \( \alpha > 0 \) and for all \( n \). It is proved that Fourier series with asymptotically even and complex quasimonotone coefficients, satisfying

\[
\lim_{n \to \infty} n^{1/q} \max_{\lambda \in [\lambda_n]} |\Delta \hat{f}(j)|^{1/q} \max_{\lambda \in [\lambda_n]} |\hat{f}(j)|^{1/p} = o(1).
\]

converges in \( L^p(T) \)-norm if and only if \( \hat{f}(n) \lg |n| = o(1) \), \( n \to \infty \). A recent result of Č. V. Stanojević [3] is a special case of the corollary of the main theorem.

1. Introduction. Recently Č. V. Stanojević [1] introduced a Tauberian \( L^1 \)-convergence class of Fourier series

\[
S[f] \sim \sum_{|n| < \infty} \hat{f}(n)e^{int}.
\]

The sequence of Fourier coefficients \( \{ \hat{f}(n) \} \) belongs to the class \( \mathcal{F} \) if for some \( 1 < p \leq 2 \)

\[
\lim_{\lambda \to 1+0} \lim_{n \to \infty} \left( \sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p} = 0.
\]

For \( \{ \hat{f}(n) \} \in \mathcal{F} \) and \( \{ \hat{f}(n) \} \) asymptotically even, i.e.

\[
\frac{1}{n} \sum_{k=1}^{n} |\hat{f}(k) - \hat{f}(-k)| \lg k = o(1), \quad n \to \infty,
\]

\[
\lim_{\lambda \to 1+0} \lim_{n \to \infty} \sum_{j=n}^{[\lambda n]} |\Delta (\hat{f}(j) - \hat{f}(-j))| \lg j = 0,
\]

it is shown in [1] that the Fourier series (1.1) converges in \( L^1(T) \)-norm, where \( T = \mathbb{R}/2\pi \mathbb{Z} \), if and only if

\[
\| \hat{f}(n)E_n + \hat{f}(-n)E_{-n} \|_{L^1(T)} = o(1), \quad n \to \infty,
\]
where

$$E_n(t) = \sum_{k=0}^{n} e^{ikt}.$$ 

It was observed in [1] that \( \{\hat{f}(n)\} \in \mathcal{C}\mathcal{C}\) implies that

\[
(\text{BOX}) \quad n^{1/(2\alpha)} \hat{f}(n) = o(1), \quad n \to \infty, \quad \frac{1}{\rho} + \frac{1}{\alpha} = 1.
\]

Proposition 3.1 in [1] and further results of W. O. Bray and Č. V. Stanojević [2] indicate that a stronger form of (BOX) together with some additional assumptions about the speed of \( \{\hat{f}(n)\} \) can be used to obtain new classes of \( L^1 \)-convergence. However, it is natural to ask are there some reasonable regularity conditions for \( \{\hat{f}(n)\} \) such that a slightly stronger form of (BOX) would imply that (ST) is a necessary and sufficient condition for \( L^1 \)-convergence of (1.1).

In this paper we shall show that it is possible to define monotonicity for a sequence of complex numbers, apply that definition to \( \{\hat{f}(n)\} \in \mathcal{C}\mathcal{C}\) and obtain some (BOX)-like conditions. It will be also shown that a result of Č. V. Stanojević [3] (Corollary 2.2) is a special case of a corollary to our main result.

2. Definitions and lemmas. In this section we shall define quasimonotonicity of a sequence of complex numbers \( \{c_n\} \) by restricting the range of the sequence \( \{\Delta c_n + \alpha c_n/n\} \) to a certain cone in the complex plane.

**Definition 2.1.** A sequence of complex numbers \( \{c_n\}_{n=1}^{\infty} \) is complex quasimonotone if there exists a cone

\[
K_{\alpha}(\theta_0) = \left\{ z \mid |\arg z| < \theta_0 \leq \frac{\pi}{2} \right\}
\]

such that for some \( \alpha > 0 \)

\[
\Delta c_n + \frac{\alpha}{n} c_n \in K_{\alpha}(\theta_0)
\]

for all \( n \).

The following lemma gives an estimate of \( \sum_{j=n}^{m} |\Delta c_j| \) that we need in our main result.

**Lemma 2.1.** Let \( \{c_n\}_{n=1}^{\infty} \) be a complex quasimonotone sequence. Then

\[
\sum_{j=n}^{m-1} |\Delta c_j| \leq \frac{|c_m - c_n|}{\cos \theta_0} + \alpha \left( 1 + \frac{1}{\cos \theta_0} \right) \sum_{j=n}^{m} \frac{|c_j|}{j}.
\]

**Proof.** To the first term of the right-hand side of the inequality

\[
\sum_{j=n}^{m-1} |\Delta c_j| \leq \sum_{j=n}^{m-1} |\Delta c_j + \frac{\alpha}{j} c_j| + \alpha \sum_{j=n}^{m} \frac{|c_j|}{j},
\]

we apply the inequality of M. Petrović [4], i.e.

\[
\sum_{j=n}^{m-1} \left| \Delta c_j + \frac{\alpha}{j} c_j \right| \leq \frac{1}{\cos \theta_0} \left| \sum_{j=n}^{m-1} \left( \Delta c_j + \frac{\alpha}{j} c_j \right) \right|.
\]
and obtain
\[
\sum_{j=n}^{m-1} |\Delta c_j| \leq \frac{1}{\cos \theta_0} \left| \sum_{j=n}^{m-1} \Delta c_j \right| + \frac{\alpha}{\cos \theta_0} \sum_{j=n}^{m} \frac{|c_j|}{j} + \alpha \sum_{j=n}^{m} \frac{|c_j|}{j}.
\]

Finally
\[
\sum_{j=n}^{m-1} |\Delta c_j| \leq \frac{1}{\cos \theta_0} |c_m - c_n| + \alpha \left( \frac{1}{\cos \theta_0} + 1 \right) \sum_{j=n}^{m} \frac{|c_j|}{j}.
\]

In particular, for \( \alpha = 0 \), we have the following definition and the corresponding lemma.

**DEFINITION 2.2.** A sequence \( (c_n)_{n=1}^{\infty} \) of complex numbers is complex monotone if there exists a cone
\[
K(\theta_0) = \left\{ z \mid |\arg z| \leq \theta_0 < \frac{\pi}{2} \right\}
\]
such that \( \Delta c_n \in K(\theta_0) \), for every \( n \).

**LEMMA 2.2.** Let \( (c_n)_{n=1}^{\infty} \) be a complex monotone sequence. Then
\[
\sum_{j=n}^{m-1} |\Delta c_j| \leq \frac{1}{\cos \theta_0} |c_m - c_n|.
\]

From (2.2) it is clear that complex monotonically decreasing sequence \( \{c_n\} \) is of bounded variation.

To state our result in a more succinct form we shall use the following lemma from [2].

**LEMMA 2.3.** The condition (ST) is equivalent to
\[
f(n) \log |n| = o(1), \quad n \to \infty.
\]

3. **Results.** Our main result gives necessary and sufficient conditions for \( L^1 \)-convergence of Fourier series with asymptotically even coefficients and satisfying some (BOX)-like conditions.

**THEOREM 3.1.** Let
\[
S[f] \sim \sum_{|n|<\infty} \hat{f}(n) e^{int}
\]
be the Fourier series of \( f \in L^1(T) \) with asymptotically even coefficients.

If \( \{\hat{f}(n)\} \) is complex quasimonotone and if for some 1 < \( p \leq 2 \),
\[
\lim_{n \to \infty} n^{1/q} \max_{n \ll j \ll \lambda n} |\Delta \hat{f}(j)|^{1/q} \max_{n \ll j \ll \lambda n} |\hat{f}(j)|^{1/p} = o(1),
\]

\[
\lambda \to 1 + 0, \quad \frac{1}{p} + \frac{1}{q} = 1,
\]

then
\[
\|S_n(f) - f\| = o(1), \quad n \to \infty.
\]
if and only if
\[ \hat{f}(n)|\lg|n| = o(1), \quad n \to \infty. \]

**Proof.** It suffices to prove that (3.1) and complex quasimonotonicity imply

\[ (\text{HK}) \quad \lim_{\lambda \to 1^+} \lim_{n \to \infty} \left( \sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(\lambda n)|^p \right)^{1/p} = 0. \]

From

\[ \left( \sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p} \leq \max_{n \leq j \leq [\lambda n]} \left( j^{1/q} |\Delta \hat{f}(j)|^{1/q} \right) \left( \sum_{j=n}^{[\lambda n]} |\Delta \hat{f}(j)| \right)^{1/p}, \]

using Lemma 2.1 we obtain

\[ \left( \sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p} \leq \max_{n \leq j \leq [\lambda n]} \left( j^{1/q} |\Delta \hat{f}(j)|^{1/q} \right) \left( \frac{|\hat{f}([\lambda n]) - \hat{f}(n)|}{\cos \theta_0} \right)^{1/p}, \]

\[ + \alpha \left( 1 + \frac{1}{\cos \theta_0} \right) \left( \sum_{j=n}^{[\lambda n]} \frac{|\hat{f}(j)|}{j} \right)^{1/p}. \]

But both \(|\hat{f}([\lambda n])|\) and \(|\hat{f}(n)|\) are less than \(\max_{n \leq j \leq [\lambda n]} \hat{f}(j)|\), and \(\sum_{j=n}^{[\lambda n]} \frac{1}{j} \leq C \lg \lambda, \)

where \(C\) is an absolute constant.

Hence, for \(\lambda > 1\) we have

\[ \left( \sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p} \leq \left[ \frac{2}{\cos \theta_0} + \alpha \left( 1 + \frac{1}{\cos \theta_0} \right) C \lg \lambda \right]^{1/p}, \]

\[ \cdot \max_{n \leq j \leq [\lambda n]} \left( j^{1/q} |\Delta \hat{f}(j)|^{1/q} \right) \max_{n \leq j \leq [\lambda n]} |\hat{f}(j)|^{1/p}. \]

Therefore from (3.1) after taking limit superior as \(n \to \infty\) followed by the limit as \(\lambda \to 1 + 0\) the proof of the theorem is obtained.

In the case of the complex monotone coefficients we have the corresponding theorem where condition (3.1) is slightly weakened.
Theorem 3.2. Let

\[ S[f] \sim \sum_{|n|<\infty} f(n)e^{int} \]

be the Fourier series of \( f \in L^1(T) \) with asymptotically even coefficients. If \( \{\hat{f}(n)\} \) is complex monotone and if for some \( 1 < p \leq 2 \),

\[ \lim_{\lambda \to 0^+} \lim_{n \to \infty} \max_{n \leq j \leq [\lambda n]} \left( j |\Delta \hat{f}(j)\|^{1/q} |\hat{f}([\lambda n]) - \hat{f}(n)|^{1/p} \right) = 0 \]

then

\[ \|S_n(f) - f\| = o(1), \quad n \to \infty, \]

if and only if

\[ \hat{f}(n) |\log n| = o(1), \quad n \to \infty. \]

Proof. Using a similar inequality as in Theorem 3.1 we have

\[ \left( \sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)\|^p \right)^{1/p} \leq \max_{n \leq j \leq [\lambda n]} \left( j^{1/q} |\Delta \hat{f}(j)\|^{1/q} \right) \left( |\hat{f}([\lambda n]) - \hat{f}(n)|/\cos \theta_0 \right)^{1/p} \]

from which the proof of Theorem 3.2 follows.

Corollary 3.1. Let

\[ S[f] \sim \sum_{|n|<\infty} f(n)e^{int} \]

be the Fourier series of \( f \in L^1(T) \) with even coefficients. If \( \{\hat{f}(n)\} \) is complex quasimonotone and if for some \( 1 < p \leq 2 \), (3.1) holds, then (1.1) converges in \( L^1 \)-norm if and only if

\[ \hat{f}(n) |\log n| = o(1), \quad n \to \infty. \]

Since

\[ n \Delta \hat{f}(n) = o(1), \quad n \to \infty, \]

implies (3.1), it is clear that the aforementioned result of Č. V. Stanojević [3] is a special case of Corollary 3.1.

4. Additional results and remarks. The conditions (AE1) and (AE2) can be rewritten as

\[ \frac{1}{[n/l_n]} \sum_{j=n}^{n+[n/l_n]} |f(j) - f(-j)| \log j = o(1), \quad n \to \infty, \]

\[ \sum_{j=n}^{n+[n/l_n]} |\Delta(f(j) - f(-j))| \log j = o(1), \quad n \to \infty, \]

where \( l_n \to +\infty, l_n = o(n), n \to \infty \) and

\[ \left\| \sigma_n[f] - \sigma_n(f) \right\|_{l_1} = o(1), \quad n \to \infty \]
(\sigma_n(f) denotes the Fejér sums). It is easy to see that (4.3) is implied by

\[(4.4) \quad \| \sigma_n(f) - f \|_{n/2} = o(1), \quad n \to \infty. \]

W. O. Bray and Č. V. Stanojević [2] used (4.4) to obtain a result relating the smoothness of \(f\) with the smoothness of \(\{\hat{f}(n)\}\). They considered the series (1.1) with coefficients satisfying (4.1) and (4.2) with \(l_n = \| \sigma_n(f) - f \|^{-1}\) and proved that if

\[(4.5) \quad n\Delta \hat{f}(n)\| \sigma_n(f) - f \| = o(1), \quad n \to \infty, \]

then the series (1.1) converges in \(L^1\)-norm if and only if

\[(4.6) \quad \hat{f}(n)\| \sigma_n(f) - f \| = o(1), \quad n \to \infty \]

(clearly, the only interesting case is \(n\| \sigma_n(f) - f \| \to \infty, n \to \infty\)).

In the next theorem we shall improve (4.5) assuming that \(\{\hat{f}(n)\}\) is a complex quasimonotone sequence.

**Theorem 4.1.** Let

\[ S[f] \sim \sum_{|n| < \infty} \hat{f}(n)e^{int} \]

be the Fourier series of \(f \in L^1(T)\), satisfying (4.1) and (4.2) with \(l_n = \| \sigma_n(f) - f \|^{-1}\).

If for some \(1 < p < 2, (\frac{1}{p} + \frac{1}{q} = 1)\),

\[(4.7) \quad n^{1/q}|\Delta \hat{f}(n)|^{1/p} \| \sigma_n(f) - f \|^{1/p} = o(1), \quad n \to \infty, \]

then

\[ \| S_n(f) - f \| = o(1), \quad n \to \infty, \]

if and only if (4.6) holds.

**Proof.** From the basic inequality in [1] we have

\[ \| \sigma_n(f) - f \|^{1/q} \left( \sum_{j = n}^{n + [n\| \sigma_n(f) - f \|]} j^{-1/p} |\Delta \hat{f}(j)|^{1/p} \right) \]

\[ \leq An^{1/q}\| \sigma_n(f) - f \|^{1/q} \max_{n < j < n + [n\| \sigma_n(f) - f \|]} |\Delta \hat{f}(j)|^{1/q} \max_{n < j < n + [n\| \sigma_n(f) - f \|]} |\hat{f}(j)|^{1/p}, \]

where \(A\) is an absolute constant. Hence, for \([\lambda n] = n + [n\| \sigma_n(f) - f \|]\), (HK) is satisfied and the proof of the theorem follows.

Since (4.7) is implied by

\[(4.8) \quad n\Delta \hat{f}(n)|f(n)|^{q-1} = O(1), \quad n \to \infty, \]

we have a corollary to Theorem 4.1.

**Corollary 4.1.** Let

\[ S[f] \sim \sum_{|n| < \infty} \hat{f}(n)e^{int} \]

be the Fourier series of \(f \in L^1(T)\), satisfying (4.1) and (4.2) with \(l_n = \| \sigma_n(f) - f \|^{-1}\).
If for some $q \geq 2$, (4.8) holds, then
\[ \|S_n(f) - f\| = o(1), \quad n \to \infty, \]
if and only if (4.6) holds.

Thus, in the case of complex quasimonotone coefficients, we have results sharper than the corresponding results in [1 and 2].

Our final remark concerns the method (HK) used in this paper. It seems that there should be a direct proof of our Theorems 3.1 and Theorem 4.1 independent of Tauberian condition (HK).

REFERENCES


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