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FREE PRODUCTS OF ABELIAN l-GROUPS ARE CARDINALLY INDECOMPOSABLE

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ABSTRACT. We show that a well-known theorem of Baer and Levi concerning the impossibility of simultaneous decomposition of a group into a free product and a direct sum has an analogue for abelian lattice ordered groups. Specifically we prove that an abelian lattice ordered group cannot be decomposed both into a free product and into a cardinal sum.

1. Introduction and notation. Baer and Levi [**1**] proved that a group which can be decomposed into a free product cannot be decomposed into a direct sum. The objective of this paper is to establish the corresponding result for abelian l-groups (lattice ordered groups). We prove that no abelian l-group is decomposable nontrivially both into an abelian l-group free product and into a cardinal sum.

The proof of the theorm of Baer and Levi (see for example Kurosh [**11**, p. 28]) is based on the Kurosh subgroup theorem for free products and the Nielsen-Schreier theorem. As the counterparts of these two theorems are not valid for abelian l-groups, our proof is more involved and evolves along different lines. The cycle of ideas leading to the proof of the main result was originated with Weinberg's construction of free abelian l-groups in [**18**]. Subsequently, Bernau [**2**, **3**] corrected and refined some of Weinberg's results. He showed, among other things, that the free abelian l-group over a partially ordered group with a nontrivial order cannot be decomposed into a cardinal sum (see Theorem 1 below), and that the same is true for any free abelian l-group of rank greater than one. Inasmuch as any free abelian l-group is a free product of copies of $Z \boxplus Z$, our result extends the latter result. Our proof utilizes one of Bernau's aforementioned results, and a construction of abelian l-group free products in terms of free extensions of partially ordered groups as was presented by the authors in [**16**] (see Theorem 2 below).

Throughout this paper, we let $\mathcal{A}$ denote the variety of abelian l-groups.

The $\mathcal{A}$*-free product* (abelian l-group free product) of a family $(G_i \mid i \in I)$ in $\mathcal{A}$, is an l-group G in $\mathcal{A}$, denoted by

$$\bigsqcup_{i \in I} G_i,$$

together with a family of l-monomorphisms $(\alpha_i \colon G_i \to G \mid i \in I)$ such that

(i) $\bigcup_{i \in I} \alpha_i(G_i)$ generates G as an l-group;

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(ii) for every $H \in \mathcal{Q}$ and every family of l-homomorphisms $(\beta_i\colon G_i \to H \mid i \in I)$, there exists an l-homomorphism $\beta\colon G \to H$ such that $\beta_i = \beta\alpha_i$ for all $i \in I$.

Following the usual practice we shall speak of G as the $\mathcal{Q}$-free product of $(G_i \mid i \in I)$, and identify each free factor G_i with its image $\alpha_i(G_i)$ in G. In what follows, unless otherwise specified, $(G_i \mid i \in I)$ will be an arbitrary family of l-groups in $\mathcal{Q}$, and we will use G to denote the $\mathcal{Q}$-free product of this family.

Let us also recall another pertinent definition. The abelian l-group $F(P)$ is the *free extension* of the partially ordered subgroup P if

(i) the inclusion map $P \to F(P)$ is an order embedding, and P generates $F(P)$ as an l-group;

(ii) for every $H \in \mathcal{Q}$ and every o-homomorphism $\beta\colon P \to H$ there exists an l-homomorphism $\gamma\colon F(P) \to H$ extending β.

It was noted in **[18]** that $F(P)$ exists if and only if P is semiclosed.

We shall denote the *cardinal product* and *cardinal sum* of a family $(H_i \mid i \in I)$ in $\mathcal{Q}$, by

$$\prod_{i \in I} H_i$$

and

$$\boxplus_{i \in I} H_i$$

respectively. In particular, we shall write $H_1 \boxplus H_2$ for the cardinal sum of H_1 and H_2. These are the group product and sum with the pointwise order. If $H = H_1 \boxplus H_2$, then H_1 (H_2) is called a *cardinal summand* of H. An l-group H is said to be *cardinally indecomposable* if its only cardinal summands are $\{0\}$ and H. The positive cone of an l-group H is denoted by H^+.

In addition to the aforementioned papers, several other papers on free products of l-groups have been written. In particular see Franchello [7], Holland and Scrimger [**10**], Martinez [**12**, **13**], and another paper by the authors [**17**]. Free extensions and free abelian l-groups have been considered in detail in the nice papers by Bernau [**2**, **3**], Conrad [**5**, **6**], and Weinberg [**18**, **19**]. Basic information on the theory of l-groups can be found in Bigard *et al.* [**4**] and Fuchs [**8**]. General references on free products and free extensions include Grätzer [**9**] and Pierce [**14**].

2. Free products are cardinally indecomposable. We begin our quest of a proof of the main result by noting the following theorem due to Bernau [**2**].

THEOREM 1. *If P is a semiclosed partially ordered group and $P^+ \neq \{0\}$, then $F(P)$ is cardinally indecomposable.*

Theorem 1 is of fundamental importance to our proof. Its usefulness becomes more apparent when we consider the representation for free products described here. For each $i \in I$, let $\Gamma_i = \{P \mid P \text{ is a prime subgroup of } G_i\}$. Also, let

$$\Gamma = \bigcup_{i \in I} \Gamma_i$$

and consider the set Δ of all choice functions $\delta\colon I \to \Gamma$. Further let

$$\Pi = \prod_{\delta \in \Delta} f\Big(\boxplus_{i \in I} (G_i/P_{\delta(i)}) \Big)$$

and denote by ρ_δ the projection of Π onto

$$F\Big(\boxplus_{i \in I} (G_i/P_{\delta(i)}) \Big)$$

for each $\delta \in \Delta$. For each $i \in I$, let $\psi_i\colon G_i \to \Pi$ be the l-homomorphism satisfying $\rho_\delta \psi_i(g) = g + P_{\delta(i)}$, for all $g \in G_i$ and $\delta \in \Delta$. Finally, let

$$\psi\colon \bigoplus_{i \in I} G_i \to \Pi$$

be the group homomorphism extending the l-homomorphisms ψ_i. With this notation established we get the following theorem from [**16**, Theorem 2.8].

THEOREM 2. *The $\mathcal{Q}$-free product*

$$G = \bigsqcup_{i \in I} G_i$$

of the family $(G_i \mid i \in I)$ is isomorphic to the sublattice G' of Π generated by

$$\psi\Big(\bigoplus_{i \in I} G_i \Big).$$

Thus, every element of G' is an element of the form

$$\bigvee_{j \in J} \bigwedge_{k \in K} \psi(h_{jk})$$

where $\{h_{jk} \mid j \in J,\ k \in K\}$ is a finite subset of

$$\bigoplus_{i \in I} G_i.$$

We turn now to two technical lemmas, one simple and the other not so simple. Henceforth we let $G_i' = \psi(G_i) = \psi_i(G_i)$.

LEMMA 3. *If $g_i \in G_i^+ \setminus \{0\}$, $g_j \in G_j^+ \setminus \{0\}$, and $i \neq j$, then $\psi_i(g_i) \wedge \psi_j(g_j) \neq 0$ in G'.*

PROOF. Since g_i, $g_j \neq 0$, there exist primes $P_i \subseteq G_i$, $P_j \subseteq G_j$ such that $g_i \notin P_i$, $g_j \notin P_j$. But then $(g_i + P_i) \wedge (g_j + P_j) \neq 0$ in a lexicographic total ordering of $(G_i/P_i) \oplus (G_j/P_j)$. Consider an arbitrary $\delta \in \Delta$ such that $\delta(i) = P_i$ and $\delta(j) = P_j$. Then $(g_i + P_i) \wedge (g_j + P_j) \neq 0$ under some total order on

$$\bigoplus_{i \in I} (G_i/P_{\delta(i)})$$

extending the cardinal order. Hence, $(g_i + P_i) \wedge (g_j + P_j) \neq 0$ in

$$F\Big(\boxplus_{i \in I} G_i/P_{\delta(i)} \Big)$$

(see Weinberg [**18**, Theorems 2.5 and 2.6]) and hence $\psi_i(g_i) \wedge \psi_j(g_j) \neq 0$ in G'. $\square$

Let us point out here that Lemma 3 is a very special case of Proposition 3.1 in [**17**].

LEMMA 4. *Consider a fixed element $j \in I$. Let $x \in G_j'$ and let $u, v \in G'$ be such that $x = u + v$ and $u \wedge v = 0$. Then $u, v \in G_j'$ if and only if for each $\delta \in \Delta$ at least one of $\rho_\delta(u)$ and $\rho_\delta(v)$ is zero.*

PROOF. (i) Suppose $u, v \in G_j'$. For each $\delta \in \Delta$ we have $\rho_\delta(u) \wedge \rho_\delta(v) = 0$ in $G_j/P_{\delta(j)}$ so either $\rho_\delta(u) \in P_{\delta(j)}$ or $\rho_\delta(v) \in P_{\delta(j)}$. It follows that $\rho_\delta(u) = 0$ or $\rho_\delta(v) = 0$ in

$$F\Big(\underset{i \in I}{\boxplus} G_i/P_{\delta(i)}\Big).$$

(ii) Conversely, suppose that for each $\delta \in \Delta$ either $\rho_\delta(u) = 0$ or $\rho_\delta(v) = 0$. Let ρ: $G' \to G_j'$ be the unique l-epimorphism extending the identity on G_j' and the zero maps $G_i' \to G_j'$ $(i \neq j)$. We will show that $u = \rho(u)$ and $v = \rho(v)$ which will complete the proof. To begin with note that for each $z \in G'$ and each $\delta \in \Delta$, $\rho_\delta(z) = 0$ implies $\rho_\delta(\rho(z)) = 0$. Indeed let $\delta \in \Delta$ and consider the l-epimorphism

$$\sigma_j\colon F\Big(\underset{i \in I}{\boxplus} G_i/P_{\delta(i)}\Big) \to G_j/P_{\delta(j)}$$

extending the identity on $G_j/P_{\delta(j)}$ and the zero maps $G_i/P_{\delta(i)} \to G_j/P_{\delta(j)}$ $(i \neq j)$. Clearly, we have $\rho_\delta/G_j' \circ \rho = \sigma_j \circ \rho_\delta$. But if $z \in G_i'$ and $\rho_\delta(z) = 0$, then $\sigma_j(\rho_\delta(z)) = 0$ from which $\rho_\delta(\rho(z)) = 0$ follows immediately.

Now, let δ be an arbitrary element of Δ. We have $\rho_\delta(x) = \rho_\delta(u) + \rho_\delta(v)$ and $\rho_\delta(x) = \sigma_j(\rho_\delta(x)) = \rho_\delta(\rho(u)) + \rho_\delta(\rho(v))$. By assumption either $\rho_\delta(u) = 0$ or $\rho_\delta(v) = 0$. If, for instance, $\rho_\delta(u) = 0$, then $\rho_\delta(\rho(u)) = 0$ and so $\rho_\delta(v) = \rho_\delta(\rho(v))$. Thus we always have $\rho_\delta(u) = \rho_\delta(\rho(u))$ and $\rho_\delta(v) = \rho_\delta(\rho(v))$. Hence, $u = \rho(u)$ and $v = \rho(v)$ as was to be shown. $\square$

We now have the tools necessary for the proof of the main theorem.

THEOREM 5. *No abelian l-group is decomposable nontrivially both into an $\mathcal{Q}$-free product and into a cardinal sum.*

PROOF. Let $G \in \mathcal{Q}$, $G \neq \{0\}$. Assume that

$$G = \bigsqcup_{i \in I} G_i$$

for a family $(G_i \mid i \in I)$ in $\mathcal{Q}$, such that $G_i \neq \{0\}$ and $G_j \neq \{0\}$ for two distinct indices i and j in I. We will work with the isomorphic copy G' of G as described in Theorem 2, and as usual write G_i' for $\psi(G_i) = \psi_i(G_i)$.

Suppose now that $G' = H_1 \boxplus H_2$. Initially we claim that

$$\bigcup_{i \in I} G_i' \subseteq H_1 \cup H_2.$$

To see this suppose there is $j \in I$ and $x \in (G_j')^+$ such that $x \notin H_1 \cup H_2$. Then there would exist elements x_1 in $H_1^+ \setminus \{0\}$ and x_2 in $H_2^+ \setminus \{0\}$ such that $x = x_1 + x_2$. But for each $\delta \in \Delta$ we have

$$F\Big(\underset{i \in I}{\boxplus} G_i/P_{\delta(i)}\Big) = \rho_\delta(H_1) \boxplus \rho_\delta(H_2),$$

so in view of Theorem 1, $\rho_\delta(H_1) = \{0\}$ or $\rho_\delta(H_2) = \{0\}$. This means $\rho_\delta(x_1) = 0$ or $\rho_\delta(x_2) = 0$, and by virtue of Lemma 4, $x_1, x_2 \in G_j'$. Next let $y \in (G_i')^+ \setminus \{0\}$ for some $i \neq j$ in I. Decompose y as $y = y_1 + y_2$ with $y_1 \in H_1^+$ and $y_2 \in H_2^+$. Again by Lemma 4, $y_1, y_2 \in G_i'$ and not both are zero. Assuming $y_2 \neq 0$ we obtain $x_1 \wedge y_2 = 0$ which is impossible by Lemma 3. Similarly, $y_1 \neq 0$ gives a contradiction via the equation $x_2 \wedge y_1 = 0$. Thus, we conclude $x \in H_1 \cup H_2$ and

$$\bigcup_{i \in I} G_i \subseteq H_1 \cup H_2.$$

We complete the proof by once again invoking Lemma 3. Let $i \in I$ and suppose there exists $x \in G_i' \cap H_1$. Then if $i \neq j$ and $y \in G_j'$ we must have $y \in H_1$. Clearly then any $x' \in G_i'$ must also be in H_1. In any case we deduce that

$$\bigcup_{i \in I} G_i' \subseteq H_1$$

or

$$\bigcup_{i \in I} G_i' \subseteq H_2,$$

which implies $H_2 = \{0\}$ or $H_1 = \{0\}$. □

One of the more important classes of ordered systems is the variety of vector lattices. It is easy to adjust the proofs of Theorems 1 and 2 to get analogous results for vector lattices (see Bernau [**2**] and the authors' paper [**16**]), and in so doing achieve the following theorem.

THEOREM 6. *No vector lattice is decomposable nontrivially both into a vector lattice free product and into a cardinal sum.*

It is reasonable to expect that the techniques of the present paper can be adapted to yield analogous results in various classes of lattice ordered modules. However, to date the only result in this direction is found in Powell [**15**] where it is shown that the free algebras of rank greater than one in a rather large class of f-modules have no nontrivial cardinal summands.

Another challenging task is to try to establish the analogue of Theorem 5 for nonabelian varieties of l-groups. Part of the difficulty is due to the fact that there is currently no reasonable representation for these free products.

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