

A NOTE ON NONFINITELY GENERATED PROJECTIVE $\mathbf{Z}\pi$ -MODULES

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ABSTRACT. Let π be a finite group and $\mathbf{Z}\pi$ be its integral group ring. It is shown that if π is not solvable, then there exists a nonfinitely generated projective $\mathbf{Z}\pi$ -module which is not free.

Let π be a finite group and $\mathbf{Z}\pi$ be its integral group ring. In [4, Theorem 7], Swan proved that if π is solvable, then any nonfinitely generated projective $\mathbf{Z}\pi$ -module is free. In this note we observe that the converse of this result is an immediate consequence of recent work of Whitehead [5]. An alternative proof of this converse also appears in a paper by Linnell [2].

Let R be a ring with an identity. If M is an R -module, the image of the natural pairing $\text{Hom}_R(M, R) \otimes R \rightarrow R$ is an ideal $\tau(M)$ called the trace ideal of M . If P is a projective R -module, $P = \tau(P)P$. Let I denote the augmentation ideal of $\mathbf{Z}\pi$.

THEOREM. *If π is not solvable, then there exists a nonfinitely generated projective $\mathbf{Z}\pi$ -module P which is not free, and satisfies $P = IP$.*

PROOF. Since π is not solvable, $\mathbf{Z}\pi$ has a nontrivial idempotent two-sided ideal A which is contained in I [1, Theorem 2.1]. $(0) \neq A = A^2 \neq \mathbf{Z}\pi$. A is finitely generated as a right ideal. Therefore, according to Whitehead [5, Corollary 2.7], there exists a countably generated projective $\mathbf{Z}\pi$ -module P whose trace ideal $\tau(P)$ is equal to A . P is not finitely generated because otherwise $\tau(P)$ would equal $\mathbf{Z}\pi$ [1, Corollary 1.4]. P is not free because $\tau(P) \neq \mathbf{Z}\pi$. $P = \tau(P)P = AP$ is contained in IP and so $P = IP$. This completes the proof.

Combining the above results with a theorem of Roggenkamp [3] gives the tidy equivalence of the following three statements of group, ring, and module theory respectively. (1) π is solvable. (2) $\mathbf{Z}\pi$ has no nontrivial idempotent ideals. (3) Every nonfinitely generated projective $\mathbf{Z}\pi$ -module is free.

REFERENCES

1. T. Akasaki, *Idempotent ideals in integral groups rings*, J. Algebra **23** (1972), 343–345.
2. P. A. Linnell, *Nonfree projective modules for integral group rings* (to appear).
3. K. W. Roggenkamp, *Integral group rings of solvable finite groups have no idempotent ideals*, Arch. Math. (Basel) **25** (1974), 125–128.
4. R. G. Swan, *The Grothendieck ring of a finite group*, Topology **2** (1963), 85–110.
5. J. M. Whitehead, *Projective modules and their trace ideals*, Comm. Algebra **8** (1980), 1873–1901.

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