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DIEUDONNÉ-SCHWARTZ THEOREM ON BOUNDED SETS IN INDUCTIVE LIMITS. II

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ABSTRACT. The Dieudonné-Schwartz Theorem [**1**, Chapter 2, §12] has been stated for strict inductive limits. In [**3**] it has been extended to inductive limits. Here the result of [**3**] is generalized. Also, the case when each set bounded in $\operatorname{ind\,lim} E_n$ is contained, but not necessarily bounded, in some E_n is considered.

Let $E_1 \subset E_2 \subset \cdots$ be a sequence of locally convex spaces and $E = \operatorname{ind\,lim} E_n$ their inductive limit (with respect to the identity maps id: $E_n \to E_{n+1}$). The Dieudonné-Schwartz theorem states that a set $B \subset E$ is bounded if and only if it is contained and bounded in some E_n, provided that

(H-1) each E_n is closed in E_{n+1}, and

(H-2) the topology of each E_n equals the topology induced in E_n by E_{n+1}. It is convenient to introduce some further hypotheses:

(H-3) each E_n is closed in E,

(H-4) each convex and closed set in E_n is closed in E_{n+1},

(H-7) for any $n \in N$ there is $p \in N$ such that $\bar{E}_n^E \subset E_{n+p}$, where $\bar{E}_n^E$ is the closure of E_n in E,

(H-8) for any closed hyperplane F in E_n, $(E_n \setminus F) \cap \bar{F}^{E_{n+1}} = \varnothing$,

(DS) each set B bounded in E is contained in some E_n, and

(DST) each set B bounded in E is contained and bounded in some E_n.

The following implications: H-1 & 2 $\Rightarrow$ H-3, H-3 $\Rightarrow$ DS, H-4 $\Rightarrow$ DST, and H-4 $\Rightarrow$ H-3, are known, see [**1**, Chapter 2, §12; **2** and **3**].

THEOREM 1. H-7 $\Rightarrow$ DS. *If E is metrizable, the implication can be reversed.*

PROOF. Assume H-7 and existence of a set B bounded in E which is not contained in any E_n. Choose a sequence $1 = n_1 \leq n_2 \leq n_3 \leq \cdots$ such that $\bar{E}_{n_k}^E \subset E_{n_{k+1}}$ and $b_k \in B \setminus E_{n_k}$, $k \in N$.

Since $b_1 \neq 0$, there exists convex 0-nbhd G_1 in E such that $b_1 \notin G_1 + G_1$. Put $V_1 = G_1 \cap E_{n_1}$ and $W_1 = \bar{V}_1^E$. Then $W_1 \subset (G_1 + G_1) \cap E_{n_2}$ and $b_1 \notin W_1$, $\frac{1}{2}b_2 \notin W_1$. Hence there exists convex 0-nbhd G_2 in E such that $b_1, \frac{1}{2}b_2 \notin W_1 + G_2 + G_2$. Put $V_2 = G_2 \cap E_{n_2}$ and $W_2 = \overline{V_1 + V_2}^E$. Again $W_2 \subset (W_1 + G_2 + G_2) \cap E_{n_3}$ and b_1, $\frac{1}{2}b_2$, $\frac{1}{3}b_3 \notin W_2$, etc. When the sequence $\{W_k\}$ is constructed, then $W = \bigcup \{W_k;\, k \in N\}$ is a 0-nbhd in E which does not absorb B.

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Let $\{G_p\}$ be a nested base for the topology of E. Assume $\overline{E}_1^E$ is not contained in any E_p. Take $x_p \in \overline{E}_1^E \setminus E_p$ and $a_p > 0$ such that $a_p x_p \in G_p$, $p \in N$. Then $B = \bigcup \{a_p x_p, p \in N\}$ is bounded in E and not contained in any E_p.

LEMMA 1. H-8 $\Leftrightarrow$ *each* $g \in E_n'$ *has a continuous extension to* E_{n+1}.

PROOF. Assume H-8 and take $g \in E_n'$, $f \neq 0$. Choose $x_0 \in E_n$, $f(x_0) \neq 0$ and put $F = F^{-1}(0)$. Since, by H-8, $x_0 \notin \overline{F}^{E_{n+1}}$ there exists $g \in E_{n+1}'$ such that $g(x_0) = f(x_0)$ and $g(x) = 0$ for $x \in \overline{F}^{E_{n+1}}$, that is $g^{-1}(0) \supset F$ and g is the sought extension of g.

Let F be a closed hyperplane in E_n. Take $f \in E_n'$ such that $f^{-1}(0) = F$. If f has an extension g to E_{n+1} then for $x \in E_n \setminus G$, $g(x) = f(x) \neq 0$, and $x \notin g^{-1}(0) = \overline{F}^{E_{n+1}}$.

LEMMA 2. DS & H-8 $\Rightarrow$ *each set* $B \subset E_n$ *which is bounded in* E *is bounded in* E_n.

PROOF. Assume $B \subset E_n$, bounded in E, but not bounded in E_n. Then B is not weakly bounded in E_n and there is $f_0 \in E_n'$ (real dual) which is not bounded on B. For each $k \in N$, take $b_k \in B$, $f_0(b_k) > k$. By induction, choose $f_p \in E_{n+p}'$ so that f_p is an extension of f_{p-1}, $p \in N$. Then $\bigcup \{f_p^{-1}(-\infty, 1); p \in N\}$ is a 0-nbhd in E which does not absorb B.

From Theorem 1 and Lemmas 1 and 2 it follows that:

THEOREM 2. H-7 & 8 $\Rightarrow$ DS & H-8 $\Rightarrow$ DST.

PROPOSITION. H-4 $\Leftrightarrow$ H-3 & 8 $\Leftrightarrow$ H-1 & 8.

PROOF. Evidently the if implications hold. To complete the cycle, assume H-1 & 8. Take a set A closed and convex in E_n. Without loss of generality, we may assume $0 \in A$. Denote by g_f a continuous extension of $f \in E_n'$ to E_{n+1}. There exists $M \subset E_n'$ such that $A = \bigcap \{f^{-1}(-\infty, 1];\ f \in M\} = \bigcap \{g_f^{-1}(-\infty, 1];\ f \in M\} \cap E_n \supset \overline{A}^{E_{n+1}}$, since E_n is closed in E_{n+1}.

We have a diagram:

$$
\begin{array}{ccccccc}
 & & 3\ \&\ 8 & \Rightarrow & 7\ \&\ 8 & \Rightarrow & \mathrm{DS}\ \&\ 8 \\
 & & \Updownarrow & & \Updownarrow & & \Downarrow \\
1\ \&\ 2 & \Rightarrow & 4 & \Rightarrow & 7\ \&\ 8 & \Rightarrow & \mathrm{DST} \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & 3 & \Rightarrow & 7 & \Rightarrow & \mathrm{DS}
\end{array}
$$

The following examples will show that H-7 & 8 do not imply H-4 and DST & H-8 do not imply H-7.

EXAMPLE 1. Take a Banach space X and its proper subspace Y (with the inherited topology). Put $E_{2n-1} = X^n \times \{0\}^N$, $E_{2n} = X^n \times Y \times \{0\}^N$, $n \in N$, all with the product topology. Then $E = \bigcup \{E_n;\ n \in N\} \subset X^N$ has the topology inherited from X^N, as well as all E_n. Hence H-8 holds. Further $\overline{E_{2n}}^E = \overline{E_{2n+1}}^E = E_{2n+1}$ and H-7 holds. On the other hand, H-3 & 4 do not hold, since $\overline{E_{2n}}^{E_{2n+1}} = E_{2n+1} \neq E_{2n}$.

EXAMPLE 2. Let $\mathfrak{D}[-n, n] = \{f \in C^\infty(R);\ \operatorname{supp} f \subset [-n, n]\}$ and $\mathfrak{D} = \operatorname{ind\,lim} \mathfrak{D}[-n, n]$. For this inductive limit DST holds by Dieudonné-Schwartz Theorem. Take $\varphi \in \mathfrak{D}$, $\operatorname{supp} \varphi = [-1, 1]$, $A = \{\varphi((p+1)x/pq);\ p, q \in N\}$, and put $E_n = \operatorname{sp}(A \cup \mathfrak{D}[-n, n])$, $n \in N$, where sp stands for the span. We equip each E_n

with the topology inherited from $\mathfrak{D}$ and H-8 holds. Since $\mathfrak{D}[-n, n] \subset E_n$, DST holds for the ind lim E_n. On the other hand the closure of E_n in E contains functions $\varphi(\frac{1}{q}x)$, $q \in N$, and since $\varphi(\frac{1}{q}x) \notin E_s$, $s = 1, 2, \ldots, q-1$, H-7 does not hold.

EXAMPLE 3. Let X, Y be the same as in Example 1. Put $E_n = X^n \times Y^n$. Then $E = X^N \cap \bigcup\{E_n;\ n \in N\}$ with the topology inherited from X^N. If B is the closed unit ball in X, then $B^N \cap E$ is bounded in E but not contained in any E_n. Hence DS and H-3 & 7 do not hold. Further $\overline{E_n}^{E_{n+1}} = E_{n+1}$ and H-1 & 4 do not hold, either. On the other hand, H-2 & 8 hold since the topology of E_n is inherited from E_{n+1}.

EXAMPLE 4. Put $W(x) = \sqrt{1 + x^2}$, $x \in (-\infty, \infty)$, and $E_n = \{f \in L^2(R);\ \|f\|^2 = \int_R |W^{-n}f|^2\,dx < +\infty\}$. The norm $\|\cdot\|_n$ makes E_n into a Hilbert space. Since the set $\mathfrak{D}$ from Example 2 is dense in each E_n, we have $E_{n+p} = \overline{\mathfrak{D}}^{E_{n+p}} \subset \overline{E_n}^{E_{n+p}} \subset \overline{E_n}^{E}$ and H-1, 2, 3, 4, 7 do not hold. But, by Theorem 4 in [2], DST holds.

To show that H-8 does not hold, take $f_k = W^n \chi_{[-k,k]} \in E_n$ and put $B = \{f_k;\ k \in N\}$. Then $\|f_k\|_n^2 = 2k$ and $B \subset E_n$. Further $\|f_k\|_{n+1}^2 \leqslant \pi$ and B is bounded in E_{n+1}. If H-8 held B would be bounded in E_n, by Lemma 2, which is not true.

REFERENCES

1. J. Horváth, *Topological vector spaces and distributions*, Vol. 1, Addison-Wesley, Reading, Mass., 1966.

2. J. Kucera and K. McKennon, *Bounded sets in inductive limits*, Proc. Amer. Math. Soc. **69** (1978), 62–64.

3. ______, *Dieudonné-Schwartz Theorem on bounded sets in inductive limits*, Proc. Amer. Math. Soc. **78** (1980), 366–368.

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