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THE PARTIAL DIFFERENTIAL EQUATION $u_t + f(u)_x = -cu$

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ABSTRACT. Lax's solution formula for the equation $u_t + f(u)_x = 0$ is extended to the equation $u_t + f(u)_x = -cu$.

1. Introduction. In this paper we consider the extension of the explicit solution formula of Lax [**1**, **2**] for the initial value problem of the partial differential equation

$$u_t + f(u)_x = 0 \tag{1.1}$$

to the equation

$$u_t + f(u)_x = -cu. \tag{1.2}$$

Initial data are given by

$$u(x,0) = u_0(x). \tag{1.3}$$

Here, c is a positive constant. In Lax's argument $f(u)$ is a convex function of u which tends to plus infinity as u approaches plus or minus infinity. In our argument we need to add the following condition on $f(u)$:

$$f(u) \in c^3, \quad |f^{(3)}(u_1) - f^{(3)}(u_2)| \leq k|u_1 - u_2|, \tag{1.4}$$

where k is a Lipschitz constant.

Nishida [**3**] and Slemrod [**4**] have considered the hyperbolic system

$$w_t = u_x, \quad u_t = f(w)_x - cu. \tag{1.5}$$

Nishida has shown the existence of a global smooth solution of (1.5) if initial data are small. On the other hand Slemrod has shown the nonexistence of smooth solutions for large initial data. It would be interesting to study the equation (1.2) as the preliminary study for (1.5). Also it could be interesting to see the difference and the similarity of the solution formulae for (1.1) and (1.2).

This paper consists of two sections. In §2 we derive the solution formula for (1.2) assuming the existence of genuine, i.e. classical, solution, then we show that the formula is applicable to generalized solutions if the discontinuities of generalized solutions are shocks. In §3 we prove the converse, namely, the existence of a generalized solution whose discontinuities are shocks.

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2. Extension of Lax's formula. Lax obtained the formula to solve single conservation laws (2.4) by using the concept of potential. In his derivation the fact that characteristic curves are straight and solutions are constant along the characteristic curves played the important role. In our case characteristic curves are no longer straight and solutions are no longer constant along characteristic curves. Nevertheless we can extend the Lax's explicit representation of solution to the case when $f(u) \in C^3, f''(u) > 0$, and $|f^{(3)}(u_2) - f^{(3)}(u_1)| \leqslant k|u_2 - u_1|$.

Before we derive the formula we need some definitions.

DEFINITION 2.1. A function $u(x, t)$ which satisfies

$$\int_0^t \int_{-\infty}^{\infty} (\phi_t u + \phi_x f(u) - c\phi u)\, dx\, dt + \int_{-\infty}^{\infty} \phi u(x, 0)\, dx = 0 \tag{2.1}$$

for arbitrary $\phi \in C^\infty(x, t)$ which has compact support is called a generalized (or weak) solution to (1.2).

DEFINITION 2.2. A function $u(x, t)$ which satisfies (1.2) in the weak sense is an admissible solution if

$$a(u^-) > a(u^+) \tag{2.2}$$

at a jump discontinuity. Here u^- and u^+ are the value of $u(x, t)$ on the left and right sides of the jump discontinuity, and $a(u) = f'(u)$. If $f(u)$ is convex, (2.2) is equivalent to

$$u^- > u^+ . \tag{2.3}$$

This condition ensures that any point in the (x, t) plane can be connected to the initial line by a backward characteristic curve(s).

DEFINITION 2.3. Suppose we denote initial data by $u_0(x)$ and a solution can be expressed in the following way:

$$u(x, t) = S(t)u_0(x).$$

Then, the operators $S(t)$ form a one-parameter semigroup if they satisfy

$$S(t_1 + t_2) = S(t_1)S(t_2), \qquad t_1, t_2 \geqslant 0, S(0) = I. \tag{2.4}$$

Now we derive the formula assuming u is a genuine, i.e. classical, solution of (1.2). Suppose $u_0(x) = u(x, 0)$ is integrable. Define the integrated function $U(x, t)$ as

$$U(x, t) = \int_{-\infty}^{x} u(y, t)\, dy, \tag{2.5}$$

then

$$U_x = u. \tag{2.6}$$

Integrating (1.2) from $-\infty$ to x and using (2.6) we obtain

$$U_t + f(U_x) = -cU, \tag{2.7}$$

where we have adjusted $f(u)$ so that $f(0) = 0$. Since $f(u)$ is a convex function of u, the inequality

$$f(u) \geqslant f(v) + a(v)(u - v)$$

holds. Applying the above inequality with $U_x = u$ and any number v, we obtain

$$a(v)U_x - f(U_x) \leqslant a(v)v - f(v).$$

Substituting (2.7) into the above inequality, we see

$$U_t + cU + a(v)U_x \leqslant a(v)v - f(v). \tag{2.8}$$

Multiplying (2.8) by e^{ct} and defining V by Ue^{ct} we obtain

$$V_t + a(v)V_x \leqslant (a(v)v - f(v))e^{ct}. \tag{2.9}$$

Since in our problem the solution along the characteristic curve $dx/dt = a(u)$ has the form $u = u_0e^{-ct}$, where u_0 is some constant, and v is any number, we can restrict v to the form

$$v = v_0e^{-ct},$$

where v_0 is any constant. Denote by y the point where the characteristic curve $dx/dt = a(v_0e^{-ct})$ through x, t intersects the initial line, then we have

$$x - y = \int_0^t a(v_0e^{-cs})\,ds. \tag{2.10}$$

Integrating (2.9) along (2.10) with $v = v_0e^{-ct}$ from 0 to t we obtain, for $t \geqslant 0$,

$$V(x,t) \leqslant V(y,0) + \int_0^t \{a(v_0e^{-cs})v_0e^{-cs} - f(v_0e^{-cs})\}e^{cs}\,ds,$$

which implies

$$U(x,t) \leqslant U(y,0)e^{-ct} + (1/c)\left[f(v_0)e^{-ct} - f(v_0e^{-ct})\right]. \tag{2.11}$$

Denote the solution to (2.10) with respect to v_0 by $z(x-y,t)$ and replace v_0 in (2.11) by z. (We shall show that in (2.10) $(x-y)$ and v_0 are one-to-one after the derivation.) We also define $g(x-y,t)$ by

$$g(x-y,t) = (1/c)\left[f(z)e^{-ct} - f(ze^{-ct})\right],$$

and $G(x,y,t)$ by

$$G(x,y,t) = U(y,0)e^{-ct} + g(x-y,t).$$

Then (2.11) becomes

$$U(x,t) \leqslant G(x,y,t). \tag{2.12}$$

This inequality holds for all choices of y. The question is when the equality holds. Compare the form of u and v, i.e., $u = u_0e^{-ct}$ and $v = v_0e^{-ct}$. We see that if $u_0 = v_0$, the sign of equality holds in (2.9) along the whole characteristic curve $dx/dt = a(u_0e^{-ct})$. In terms of y, for the value for which $z(x-y,t)$ is equal to u, the equality holds in (2.9) along the whole characteristic curve issuing from (x,t), and therefore equality also holds in (2.11). Equivalently the equality holds in (2.12) for the value of y which minimizes $G(x,y,t)$. In the following lemmas first, we show that (2.10) is solvable with respect to v_0, and then prove that the minimum of $G(x,y,t)$ is attainable.

LEMMA 2.1. *$z(x-y,t)$ and $(x-y)$ are one-to-one for each t.*

PROOF. It is obvious that for each v_0 there exists y such that (2.10) is satisfied. To see $z(x-y,t)$ and $(x-y)$ are one-to-one for each t, substitute $z(x-y,t)$ into (2.10) and differentiate (2.10) with respect to $(x-y)$. Then we obtain

$$z_1 = 1 \Big/ \int_0^t a'(ze^{-cs})e^{-cs}\,ds, \tag{2.13}$$

where z_1 is the first derivative of z with respect to $(x-y)$. Since $a'(u)$ is positive, the denominator is positive. This means $z(x-y,t)$ is a monotone function of $(x-y)$ for each t, so that $z(x-y,t)$ and $(x-y)$ are one-to-one for each t. Q.E.D.

LEMMA 2.2. *Suppose $f(u)$ is a strictly convex function which tends to plus infinity as u approaches plus or minus infinity. Then, for each fixed t, $g(x-y,t)$ has only one minimum and tends to plus infinity as u tends to plus or minus infinity.*

PROOF. If we differentiate $g(x-y,t)$ with respect to the first argument, we have

$$\begin{aligned} g_1(x-y,t) &= (1/c)[f'(z) - f'(ze^{-ct})]z_1e^{-ct} \\ &= (1/c)f''(p)zz_1(1-e^{-ct})e^{-ct}, \end{aligned} \tag{2.14}$$

where g_1 and z_1 denote the derivative of g and z with respect to the first argument and p is between z and ze^{-ct}. From (2.13) z_1 is positive, and z is a monotonically increasing function of $(x-y)$ which tends to plus of minus infinity as $(x-y)$ approaches the endpoints of its definition. Hence $g(x-y,t)$ has only one minimum for each t.

Next, rewrite $g(x-y,t)$ in the following way:

$$\begin{aligned} g(x-y,t) &= (1/c)\{f(z)e^{-ct} - f(ze^{-ct})e^{-ct} - f(ze^{-ct})(1-e^{-ct})\} \\ &= (1/c)(1-e^{-ct})\{a(p)ze^{-ct} - f(ze^{-ct})\}, \end{aligned} \tag{2.15}$$

where p is between z and ze^{-ct}. Set $w = ze^{-ct}$ and denote the quantity inside the brackets in (2.15) by K. Then if $z > 0$, $z > p > ze^{-ct} > 0$, so that

$$K = a(p)w - f(w) > a(w)w - f(w).$$

Also if $z < 0$, $0 > ze^{-ct} > p > z$, so that

$$K > a(w)w - f(w).$$

Suppose we set $b = a(w)$ and define $h(b)$ by

$$h(b) = a(w)w - f(w).$$

Since a is monotone, w tends to plus (or minus) infinity as b approaches the upper (or lower) endpoint of the range of a. Differentiate $h(b)$, then we obtain

$$\frac{dh}{db} = w\frac{da}{dw}\cdot\frac{dw}{db} + a(w)\frac{dw}{db} - a(w)\frac{dw}{db} = w.$$

We see $h(b)$ tends to plus infinity as w approaches plus or minus infinity. Hence, for each t, K tends to plus infinity as z approaches plus or minus infinity. This completes the lemma. Q.E.D.

Indeed, equation (2.7) is the integral form of (1.2), so that inequality (2.12) is also valid for generalized solutions. Suppose all discontinuities are shocks. In this case every point can be connected to a point y on the initial line by a backward characteristic curve, and the sign of equality holds in (2.12) for that value of y. Thus, the above result is applicable to generalized solutions of (1.2) whose discontinuities are shocks. We now state this result in the following theorem.

THEOREM 2.1. *Let u be a generalized solution to* (1.2) *whose discontinuities are shocks, then*

$$u(x, t) = z(x - y, t)e^{-ct}, \tag{2.16}$$

where $y = y(x, t)$ is the value which minimizes

$$G(x, y, t) = U_0(y)e^{-ct} + g(x - y, t). \tag{2.17}$$

Here, $v_0 = z(x - y, t)$ is the solution to

$$x - y = \int_0^t a(v_0 e^{-cs})\, ds, \tag{2.18}$$

g is defined by

$$g(x - y, t) = (1/c)\left[f(z)e^{-ct} - f(ze^{-ct})\right], \tag{2.19}$$

and

$$U_0(y) = \int_{-\infty}^{y} u_0(x)\, dx, \qquad u_0(x) = u(x,0).$$

Since we choose a real characteristic curve(s) and real initial data, we have the following

COROLLARY 2.1. *For any value of t (> 0) and x,*

$$|u(x, t)| \leqslant \max_{-\infty < x < \infty} |u(x,0)|e^{-ct}.$$

3. Proof of the converse. Does the formula (2.16)–(2.20) define a generalized solution whose discontinuities are only shocks, hence proving the existence of a generalized solution? Because of a technical reason we first prove the converse for the case when $f(u)$ is a convex polynomial function. (The following calculation shows we can include terms whose power of u is not integer as far as $f^{(3)}(u)$ is not singular.)

THEOREM 3.1. *Suppose $f(u)$ is a convex polynomial function of the form*

$$f(u) = a_0 u^{2n} + a_1 u^{2n-1} + \cdots + a_{2n-1} u,$$

then the formula (2.16)–(2.20) *defines a solution to* (1.2) *which possibly includes discontinuities for arbitrary integrable initial data $u_0(x)$. The function u satisfies* (1.2) *in the sense of distributions, and the discontinuities of u are shocks. Also the formula has the semigroup property.*

PROOF. The proof is almost the same as in Lax except for the proof of convexity of $g(x - y, t)$. We prove the convexity of $g(x - y, t)$ only and omit the rest of the proof.

If we differentiate $g(x - y, t)$ twice with respect to the first argument and calculate the first and second derivatives of z with respect to the first argument by differentiating (2.19), we obtain

$$g_{11}(x-y,t)$$
$$= \frac{1}{cA^2} e^{-ct}\{2na_0 z^{2n-2}(1-e^{-(2n-1)ct})$$
$$+ (2n-1)a_1 z^{2n-3}(1-e^{-(2n-2)ct}) + \cdots + 2a_{2n-2}(1-e^{-ct})\},$$

where

$$A = \int_0^t a'(ze^{-cs})e^{-cs}\,ds \qquad (>0).$$

Define the quantity inside the braces by Q. The form of Q suggests the differentiation of Q along the line (2.18). This is always possible because from Lemma 2.1 we see that for each x and t we have at least one value of y at which G attains its minimum. Since $z(x-y,t)$ is constant along the line (2.18),

$$\frac{dQ}{dt} = cf''(ze^{-ct})e^{-ct} > 0.$$

This means dQ/dt is an increasing function along (2.18). As $Q=0$ at $t=0$, Q is positive for $t>0$. Hence, we see that for each t, $g_{11}(x-y,t)$ is positive for all x. Q.E.D.

Theorem 3.1 suggests the possibility of extending the result to a general convex function $f(u)$ by approximating f by a convex polynomial function. Indeed if the domain of it is bounded, we can apply the following lemma in Shisha [**5**].

LEMMA 3.1. *Let f be a real function which satisfies the conditions*

$$f''(x) > 0, \tag{3.1}$$

$$|f^{(3)}(x_2) - f^{(3)}(x_1)| \leq k|x_2 - x_1| \tag{3.2}$$

throughout $[a,b]$, *where k is a positive constant. Then for every integer n* (≥ 3) *there exists a convex polynomial $p(x)$ of degree $\leq n$ such that*

$$p''(x) > f''(x),$$
$$\max|f^{(3)}(x) - p^{(3)}(x)| \leq k\pi(b-a)/4(n-2),$$
$$\max|f^{(i)}(x) - p^{(i)}(x)| \leq k\pi(b-a)^{4-i}/2(n-2), \qquad i=0,1,2.$$

Now the task is to bound the domain of f. Assume $u_0(x)$ is bounded on $(-\infty,\infty)$. Since $g(x-y,t)$ tends to plus infinity as $(x-y)$ approaches the endpoints of the domain of definition, if we take sufficiently large M (>0), $z(x-y,t)\in[-M,M]$ for arbitrary t. Then we can approximate $f(u)$ by the above $p(u)$. This result can be stated in the following theorem.

THEOREM 3.2. *Suppose $f(u)$ satisfy* (3.1) *and* (3.2). *$u_0(x)$ is bounded and integrable. Then the formula* (2.16)–(2.20) *defines a solution possibly including discontinuities. The function* (2.16) *satisfies* (1.2) *in the sense of distributions, and the discontinuities of u are shocks.*

PROOF. If we assume all discontinuities are shocks, then we can derive the formula for $f(u)$ which satisfies (3.1) and (3.2). Since $g(x - y, t)$ is a convex function of $(x - y)$ which tends to plus infinity as $(x - y)$ approaches the endpoints of its definition and the test functions for weak solutions have compact supports, it is always possible to bound the value of x and y so that the minimum (or minima) of $G(x, y, t)$ is inside the bound of x and y. Since x and y are bounded, $z(x - y, t)$ is bounded. Denote the upper bound and the lower bound by z_U and z_L, respectively. Then we can approximate $f(u)$ for $u \in [z_L, z_U]$ by a polynomial $p(u)$ in Lemma 3.1. Since the approximation is uniform, in the limit when we take the degree of $p(u)$ to be infinity the converse holds for $f(u)$. Q.E.D.

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