PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 3, November 1982

DECOMPOSABLE POSITIVE MAPS ON C^*-ALGEBRAS

ERLING STØRMER

ABSTRACT. It is shown that a positive linear map of a C^*-algebra A into $B(H)$ is decomposable if and only if for all $n \in \mathbf{N}$ whenever (x_{ij}) and (x_{ji}) belong to $M_n(A)^+$ then $(\phi(x_{ij}))$ belongs to $M_n(B(H))^+$.

A positive linear map ϕ of a C^*-algebra A into $B(H)$—the bounded linear operators on a complex Hilbert space H—is said to be decomposable if there are a Hilbert space K, a bounded linear operator v of H into K, and a Jordan homomorphism π of A into $B(K)$ such that $\phi(x) = v^*\pi(x)v$ for all $x \in A$. Such maps have been studied in [**2**, **3**, **5**, **7**, **8**, **9**], and are the natural symmetrization of the completely positive ones, defined as those ϕ as above with π a homomorphism. If $M_n(B)$ denotes the $n \times n$ matrices over a subspace B of a C^*-algebra and $M_n(B)^+$ the positive part of $M_n(B)$, the celebrated Stinespring theorem [**4**] states that a map ϕ: $A \to B(H)$ is completely positive if and only if for all $n \in \mathbf{N}$ whenever $(x_{ij}) \in M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$. It is the purpose of the present note to provide an analogous characterization of decomposable maps.

THEOREM. *Let A be a C^*-algebra and ϕ a linear map of A into $B(H)$. Then ϕ is decomposable if and only if for all $n \in \mathbf{N}$ whenever (x_{ij}) and (x_{ji}) belong to $M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$.*

PROOF. Suppose ϕ is decomposable, so of the form $v^*\pi v$. If π is a homomorphism (resp. antihomomorphism) and (x_{ij}) (resp. (x_{ji})) belongs to $M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$. Since every Jordan homomorphism is the sum of a homomorphism and an antihomomorphism [**6**], if both (x_{ij}) and (x_{ji}) belong to $M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$.

Conversely suppose (x_{ij}) and $(x_{ji}) \in M_n(A)^+$ implies $(\phi(x_{ij})) \in M_n(B(H))^+$ for all $n \in \mathbf{N}$. Since this property persists when ϕ is extended to the second dual of A we may assume A is unital and that $A \subset B(L)$ for some Hilbert space L. Let t denote the transpose map on $B(L)$ with respect to some orthonormal basis. Let

$$V = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^t \end{pmatrix} \in M_2(B(L)) : x \in A \right\}.$$

Then V is a selfadjoint subspace of $M_2(B(L))$ containing the identity. Define θ_n on $M_n(B(L))$ by $\theta_n((x_{ij})) = (x_{ji}^t)$. Then θ is an antiautomorphism of order 2. Hence if

Received by the editors September 25, 1981 and, in revised form, December 4, 1981.
1980 *Mathematics Subject Classification*. Primary 46L05; Secondary 46L50.
Key words and phrases. Positive maps.

$(x_{ij}) \in M_n(A)$ then $(x_{ji}) \in M_n(A)^+$ if and only if $(x_{ij}^t) = \theta_n((x_{ji})) \in M_n(B(L))^+$. Therefore both (x_{ij}) and (x_{ji}) belong to $M_n(A)^+$ if and only if

$$\left(\begin{pmatrix} x_{ij} & 0 \\ 0 & x_{ij}^t \end{pmatrix}\right) \in M_n(V)^+ .$$

Let $\bar{\phi}$: $V \to B(H)$ be defined by

$$\bar{\phi}\left(\begin{pmatrix} x & 0 \\ 0 & x^t \end{pmatrix}\right) = \phi(x).$$

Then $\bar{\phi}$ is completely positive in the sense of [**1**] by our hypothesis on ϕ and the above equivalence. By Arveson's extension theorem [**1**, Theorem 1.2.3] $\bar{\phi}$ has an extension to a completely positive map $\bar{\bar{\phi}}$: $M_2(B(L)) \to B(H)$. By Stinespring's theorem [**4**] there are a Hilbert space K, a bounded linear map v of H into K, and a representation π_1 of $M_2(B(L))$ on K such that $\bar{\bar{\phi}} = v^*\pi_1 v$. Let π_2 be the Jordan homomorphism of A into $M_2(B(L))$ defined by

$$\pi_2(x) = \begin{pmatrix} x & 0 \\ 0 & x^t \end{pmatrix}, \qquad x \in A.$$

Then $\pi = \pi_1 \circ \pi_2$ is a Jordan homomorphism of A into $B(K)$ such that $\phi(x) = v^*\pi(x)v$ for all $x \in A$, hence ϕ is decomposable. The proof is complete.

The first example of a nondecomposable positive map was exhibited by Choi [**2**]. An extension of his example was reproduced in [**3**] together with a complete proof based on nontrivial results on biquadratic forms. We conclude by giving a short proof of his result. The example is ϕ: $M_3(\mathbf{C}) \to M_3(\mathbf{C})$ defined by

$$\phi\left(\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}\right) = \begin{pmatrix} \alpha_{11} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_{22} & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_{33} \end{pmatrix} + \mu \begin{pmatrix} \alpha_{33} & 0 & 0 \\ 0 & \alpha_{11} & 0 \\ 0 & 0 & \alpha_{22} \end{pmatrix},$$

where $\mu \geqslant 1$. It was shown by Choi that ϕ is positive. We show ϕ is not decomposable. Let $(x_{ij}) \in M_3(M_3(\mathbf{C}))$ be the matrix:

$$(x_{ij}) = \left(\begin{array}{ccc|ccc|ccc} 2\mu & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 2\mu \\ 0 & 4\mu^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2\mu & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 2\mu \\ 0 & 0 & 0 & 0 & 0 & 4\mu^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 4\mu^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2\mu & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 2\mu \end{array}\right)$$

Then both (x_{ij}) and (x_{ji}) belong to $M_3(M_3(\mathbf{C}))^+$ while it is easily seen that the matrix $(\phi(x_{ij}))$ is not positive. Hence ϕ is not decomposable by the theorem.

REFERENCES

1. W. Arveson, *Subalgebras of C^*-algebras*, Acta Math. **123** (1969), 141–224.
2. M. D. Choi, *Positive semidefinite biquadratic forms*, Linear Algebra and Appl. **12** (1975), 95–100.
3. ______, *Some assorted inequalities for positive linear maps on C^*-algebras*, J. Operator Theory **4** (1980), 271–285.
4. W. F. Stinespring, *Positive functions on C^*-algebras*, Proc. Amer. Math. Soc. **6** (1955), 211–216.
5. E. Størmer, *Positive linear maps of operator algebras*, Acta Math. **110** (1963), 233–278.
6. ______, *On the Jordan structure of C^*-algebras*, Trans. Amer. Math. Soc. **120** (1965), 438–447.
7. ______, *Decomposition of positive projections on C^*-algebras*, Math. Ann. **27** (1980), 21–41.
8. S. L. Woronowicz, *Nonextendible positive maps*, Comm. Math. Phys. **51** (1976), 243–282.
9. ______, *Positive maps of low dimensional matrix algebras*, Rep. Math. Phys. **10** (1976), 165–183.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, OSLO, NORWAY