

DECOMPOSABLE POSITIVE MAPS ON C^* -ALGEBRAS

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ABSTRACT. It is shown that a positive linear map of a C^* -algebra A into $B(H)$ is decomposable if and only if for all $n \in \mathbb{N}$ whenever (x_{ij}) and (x_{ji}) belong to $M_n(A)^+$ then $(\phi(x_{ij}))$ belongs to $M_n(B(H))^+$.

A positive linear map ϕ of a C^* -algebra A into $B(H)$ —the bounded linear operators on a complex Hilbert space H —is said to be decomposable if there are a Hilbert space K , a bounded linear operator v of H into K , and a Jordan homomorphism π of A into $B(K)$ such that $\phi(x) = v^*\pi(x)v$ for all $x \in A$. Such maps have been studied in [2, 3, 5, 7, 8, 9], and are the natural symmetrization of the completely positive ones, defined as those ϕ as above with π a homomorphism. If $M_n(B)$ denotes the $n \times n$ matrices over a subspace B of a C^* -algebra and $M_n(B)^+$ the positive part of $M_n(B)$, the celebrated Stinespring theorem [4] states that a map $\phi: A \rightarrow B(H)$ is completely positive if and only if for all $n \in \mathbb{N}$ whenever $(x_{ij}) \in M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$. It is the purpose of the present note to provide an analogous characterization of decomposable maps.

THEOREM. *Let A be a C^* -algebra and ϕ a linear map of A into $B(H)$. Then ϕ is decomposable if and only if for all $n \in \mathbb{N}$ whenever (x_{ij}) and (x_{ji}) belong to $M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$.*

PROOF. Suppose ϕ is decomposable, so of the form $v^*\pi v$. If π is a homomorphism (resp. antihomomorphism) and (x_{ij}) (resp. (x_{ji})) belongs to $M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$. Since every Jordan homomorphism is the sum of a homomorphism and an antihomomorphism [6], if both (x_{ij}) and (x_{ji}) belong to $M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$.

Conversely suppose (x_{ij}) and $(x_{ji}) \in M_n(A)^+$ implies $(\phi(x_{ij})) \in M_n(B(H))^+$ for all $n \in \mathbb{N}$. Since this property persists when ϕ is extended to the second dual of A we may assume A is unital and that $A \subset B(L)$ for some Hilbert space L . Let t denote the transpose map on $B(L)$ with respect to some orthonormal basis. Let

$$V = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^t \end{pmatrix} \in M_2(B(L)): x \in A \right\}.$$

Then V is a selfadjoint subspace of $M_2(B(L))$ containing the identity. Define θ_n on $M_n(B(L))$ by $\theta_n((x_{ij})) = (x'_{ji})$. Then θ is an antiautomorphism of order 2. Hence if

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$(x_{ij}) \in M_n(A)$ then $(x_{ji}) \in M_n(A)^+$ if and only if $(x'_{ij}) = \theta_n((x_{ji})) \in M_n(B(L))^+$. Therefore both (x_{ij}) and (x_{ji}) belong to $M_n(A)^+$ if and only if

$$\left(\begin{pmatrix} x_{ij} & 0 \\ 0 & x'_{ij} \end{pmatrix} \right) \in M_n(V)^+.$$

Let $\bar{\phi}: V \rightarrow B(H)$ be defined by

$$\bar{\phi} \left(\begin{pmatrix} x & 0 \\ 0 & x' \end{pmatrix} \right) = \phi(x).$$

Then $\bar{\phi}$ is completely positive in the sense of [1] by our hypothesis on ϕ and the above equivalence. By Arveson's extension theorem [1, Theorem 1.2.3] $\bar{\phi}$ has an extension to a completely positive map $\underline{\bar{\phi}}: M_2(B(L)) \rightarrow B(H)$. By Stinespring's theorem [4] there are a Hilbert space K , a bounded linear map v of H into K , and a representation π_1 of $M_2(B(L))$ on K such that $\underline{\bar{\phi}} = v^* \pi_1 v$. Let π_2 be the Jordan homomorphism of A into $M_2(B(L))$ defined by

$$\pi_2(x) = \begin{pmatrix} x & 0 \\ 0 & x' \end{pmatrix}, \quad x \in A.$$

Then $\pi = \pi_1 \circ \pi_2$ is a Jordan homomorphism of A into $B(K)$ such that $\phi(x) = v^* \pi(x) v$ for all $x \in A$, hence ϕ is decomposable. The proof is complete.

The first example of a nondecomposable positive map was exhibited by Choi [2]. An extension of his example was reproduced in [3] together with a complete proof based on nontrivial results on biquadratic forms. We conclude by giving a short proof of his result. The example is $\phi: M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C})$ defined by

$$\phi \left(\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \right) = \begin{pmatrix} \alpha_{11} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_{22} & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_{33} \end{pmatrix} + \mu \begin{pmatrix} \alpha_{33} & 0 & 0 \\ 0 & \alpha_{11} & 0 \\ 0 & 0 & \alpha_{22} \end{pmatrix},$$

where $\mu \geq 1$. It was shown by Choi that ϕ is positive. We show ϕ is not decomposable. Let $(x_{ij}) \in M_3(M_3(\mathbb{C}))$ be the matrix:

$$(x_{ij}) = \begin{array}{|ccc|ccc|ccc|} \hline 2\mu & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 2\mu \\ 0 & 4\mu^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2\mu & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 2\mu \\ 0 & 0 & 0 & 0 & 0 & 4\mu^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 4\mu^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2\mu & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 2\mu \\ \hline \end{array}$$

Then both (x_{ij}) and (x_{ji}) belong to $M_3(M_3(\mathbb{C}))^+$ while it is easily seen that the matrix $(\phi(x_{ij}))$ is not positive. Hence ϕ is not decomposable by the theorem.

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