DECOMPOSABLE POSITIVE MAPS ON C*-ALGEBRAS

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Abstract. It is shown that a positive linear map of a C*-algebra $A$ into $B(H)$ is decomposable if and only if for all $n \in \mathbb{N}$ whenever $(x,\gamma)$ and $(\gamma,\beta)$ belong to $M_n(A)^+$ then $(\phi(x,\gamma))$ belongs to $M_n(B(H))^+$.

A positive linear map $\phi$ of a C*-algebra $A$ into $B(H)$—the bounded linear operators on a complex Hilbert space $H$—is said to be decomposable if there are a Hilbert space $K$, a bounded linear operator $v$ of $H$ into $K$, and a Jordan homomorphism $\tau$ of $A$ into $B(K)$ such that $\phi(x) = v^{*}\tau(x)v$ for all $x \in A$. Such maps have been studied in [2, 3, 5, 7, 8, 9], and are the natural symmetrization of the completely positive ones, defined as those $\phi$ as above with $\tau$ a homomorphism. If $M_n(B)$ denotes the $n \times n$ matrices over a subspace $B$ of a C*-algebra and $M_n(B)^+$ the positive part of $M_n(B)$, the celebrated Stinespring theorem [4] states that a map $\phi: A \rightarrow B(H)$ is completely positive if and only if for all $n \in \mathbb{N}$ whenever $(x_{ij}) \in M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$. It is the purpose of the present note to provide an analogous characterization of decomposable maps.

Theorem. Let $A$ be a C*-algebra and $\phi$ a linear map of $A$ into $B(H)$. Then $\phi$ is decomposable if and only if for all $n \in \mathbb{N}$ whenever $(x_{ij})$ and $(x_{ji})$ belong to $M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$.

Proof. Suppose $\phi$ is decomposable, so of the form $v^{*}\tau v$. If $\tau$ is a homomorphism (resp. antihomomorphism) and $(x_{ij})$ (resp. $(x_{ji})$) belongs to $M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$. Since every Jordan homomorphism is the sum of a homomorphism and an antihomomorphism [6], if both $(x_{ij})$ and $(x_{ji})$ belong to $M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$.

Conversely suppose $(x_{ij})$ and $(x_{ji}) \in M_n(A)^+$ implies $(\phi(x_{ij})) \in M_n(B(H))^+$ for all $n \in \mathbb{N}$. Since this property persists when $\phi$ is extended to the second dual of $A$ we may assume $A$ is unital and that $A \subset B(L)$ for some Hilbert space $L$. Let $t$ denote the transpose map on $B(L)$ with respect to some orthonormal basis. Let

$$V = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^* \end{pmatrix} \in M_2(B(L)): x \in A \right\}.$$ 

Then $V$ is a selfadjoint subspace of $M_2(B(L))$ containing the identity. Define $\theta_n$ on $M_n(B(L))$ by $\theta_n(x_{ij}) = (x_{ji})$. Then $\theta$ is an antiautomorphism of order 2. Hence
Let $\phi: V \to B(H)$ be defined by

$$\phi\left(\begin{pmatrix} x & 0 \\ 0 & x' \end{pmatrix}\right) = \phi(x).$$

Then $\phi$ is completely positive in the sense of [1] by our hypothesis on $\phi$ and the above equivalence. By Arveson’s extension theorem [1, Theorem 1.2.3] $\phi$ has an extension to a completely positive map $\overline{\phi}: M_2(B(L)) \to B(H)$. By Stinespring’s theorem [4] there is a Hilbert space $K$, a bounded linear map $v$ of $H$ into $K$, and a representation $\pi_1$ of $M_2(B(L))$ on $K$ such that $\overline{\phi} = v^*\pi_1v$. Let $\pi_2$ be the Jordan homomorphism of $A$ into $M_2(B(L))$ defined by

$$\pi_2(x) = \begin{pmatrix} x & 0 \\ 0 & x' \end{pmatrix}, \quad x \in A.$$

Then $\tau = \pi_1 \circ \pi_2$ is a Jordan homomorphism of $A$ into $B(K)$ such that $\phi(x) = v^*\pi(x)v$ for all $x \in A$, hence $\phi$ is decomposable. The proof is complete.

The first example of a nondecomposable positive map was exhibited by Choi [2]. An extension of his example was reproduced in [3] together with a complete proof based on nontrivial results on biquadratic forms. We conclude by giving a short proof of his result. The example is $\phi: M_3(C) \to M_3(C)$ defined by

$$\phi\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} \alpha_{11} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_{22} & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_{33} \end{pmatrix} + \mu \begin{pmatrix} \alpha_{33} & 0 & 0 \\ 0 & \alpha_{11} & 0 \\ 0 & 0 & \alpha_{22} \end{pmatrix},$$

where $\mu \gg 1$. It was shown by Choi that $\phi$ is positive. We show $\phi$ is not decomposable. Let $(x_{ij}) \in M_3(M_3(C))$ be the matrix:

$$\begin{pmatrix} 2\mu & 0 & 0 & 0 & 2\mu & 0 & 0 & 2\mu \\ 0 & 4\mu^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then both $(x_{ij})$ and $(x''_{ij})$ belong to $M_3(M_3(C))^+$ while it is easily seen that the matrix $(\phi(x_{ij}))$ is not positive. Hence $\phi$ is not decomposable by the theorem.
REFERENCES
2. M. D. Choi, Positive semidefinite biquadratic forms, Linear Algebra and Appl. 12 (1975), 95–100.

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