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DISCRETE GENERALIZED CESÀRO OPERATORS

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ABSTRACT. For $|\lambda| \leq 1$, A_λ^* is the operator defined formally on the Hardy space H^2 by

$$(A_\lambda^* f)(z) = -(\lambda - z)^{-1} \int_\lambda^z f(s)\,ds, \qquad |z| < 1.$$

If $\lambda = 1$, then the usual identification of H^2 with l^2 takes A_1 onto the discrete Cesàro operator. Here we answer questions about boundedness, spectra, unitary equivalence, compactness, and subnormality for the operators A_λ.

The Cesàro operator C_0 acting on the Hilbert space l^2 of square-summable complex sequences $\{a_n\}_{n=0}^\infty$ is defined by $C_0\{a_n\} = \{b_n\}$ where $b_n = \sum_{j=0}^n a_j/(n+1)$, $n = 0, 1, 2, \ldots$. This operator was studied extensively in [1] where it was shown, among other things, that C_0 is bounded with $\|C_0\| = 2$ and spectrum $\{z \colon |1 - z| \leq 1\}$. In [4] it was proved that C_0 is a subnormal operator.

For $0 < |\lambda| \leq 1$ we define the operator A_λ on l^2 by $A_\lambda\{a_n\} = \{c_n\}$ where $c_n = \sum_{j=0}^n \bar{\lambda}^{n-j} a_j/(n+1)$, $n = 0, 1, 2, \ldots$; also define A_0 by $A_0\{a_n\} = \{a_n/(n+1)\}$. Observe that $A_1 = C_0$. We identify l^2 isometrically with the Hardy space H^2 by sending $\{a_n\}_{n=0}^\infty$ onto $f(z) = \sum_{n=0}^\infty a_n z^n$. A_λ then becomes an operator on H^2.

A_λ^* can be expressed in closed form as follows. If $f(z) = \sum_{n=0}^\infty a_n z^n$, then $(A_\lambda^* f)(z) = \sum_{k=0}^\infty c_k z^k$ where $c_k = \sum_{n=k}^\infty a_n \lambda^{n-k}/(n+1)$. Consider $\int_\lambda^z f(s)\,ds$ where the path of integration is sufficiently nice. If $\lambda = 1$ and the path consists of two segments, one connecting 1 to 0 and the other connecting 0 to z, we have $\int_1^z f(s)\,ds = \int_0^z f(s)\,ds - \int_0^1 f(s)\,ds$; the last integral exists by the Fejér-Riesz inequality [2, p. 46]. Integrating the Taylor series for f term-by-term we have $\int_\lambda^z f(s)\,ds = \sum_{n=0}^\infty a_n z^{n+1}/(n+1) - \sum_{n=0}^\infty a_n \lambda^{n+1}/(n+1)$. Comparing these Taylor coefficients with the Taylor coefficients of $(\lambda - z)(A_\lambda^* f)(z)$ we see that

$$(\lambda - z)(A_\lambda^* f)(z) = -\int_\lambda^z f(s)\,ds, \qquad |z| < 1.$$

Hence

$$(A_\lambda^* f)(z) = -(\lambda - z)^{-1} \int_\lambda^z f(s)\,ds.$$

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In matrix form, we have

$$A_\lambda^* = \begin{bmatrix} 1 & \frac{\lambda}{2} & \frac{\lambda^2}{3} & \frac{\lambda^3}{4} & \cdots \\ 0 & \frac{1}{2} & \frac{\lambda}{3} & \frac{\lambda^2}{4} & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{\lambda}{4} & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Since $\|A_\lambda f\| \leqslant \|C_0(|f|)\|$, we see that A_λ is bounded with $\|A_\lambda\| \leqslant 2$. In order to get more information, we need some lemmas. The proof of Lemma 0 is found in [**3**].

LEMMA 0 (SCHUR TEST). *If $\alpha_{ij} \geqslant 0$ $(i, j = 0, 1, 2, \ldots)$, if $p_i > 0$ $(i = 0, 1, 2, \ldots)$, and if β and γ are positive numbers such that*

$$\sum_i \alpha_{ij} p_i \leqslant \beta p_j \qquad (j = 0, 1, 2, \ldots),$$
$$\sum_j \alpha_{ij} p_j \leqslant \gamma p_i \qquad (i = 0, 1, 2, \ldots),$$

then there exists an operator A on l^2 with $\|A\|^2 \leqslant \beta\gamma$ and matrix $\langle \alpha_{ij} \rangle$ (with respect to a suitable orthonormal basis).

LEMMA 1. *Assume $0 \leqslant \alpha \leqslant 1$ and n is a positive integer. Define $B_\alpha(n) = \langle \beta_{ij} \rangle_{i,j=1}^\infty$ by*

$$\beta_{ij} = \begin{cases} \alpha^{i-j}/(i+1) & \text{if } i \geqslant j + n, \\ 0 & \text{otherwise}. \end{cases}$$

Then $B_\alpha(n)$ is bounded (on l^2) and $\|B_\alpha(n)\| \leqslant 2\alpha^n$.

PROOF. For $\alpha > 0$ we apply Lemma 0 with $p_i = \alpha^i/\sqrt{i+1}$. (The case $\alpha = 0$ is trivial.) For $i = 0, 1, 2, \ldots, n-1$, we have

$$\sum_j \beta_{ij} p_j = 0 = 0 p_i.$$

For $i \geqslant n$ we have

$$\sum_j \beta_{ij} p_j = \sum_{j=0}^{i-n} \frac{\alpha^{i-j}}{i+1} \frac{\alpha^j}{\sqrt{j+1}} \leqslant \frac{\alpha^i}{i+1} \int_0^{i-n+1} \frac{dx}{\sqrt{x}}$$
$$= \frac{\alpha^i}{i+1} 2\sqrt{i-n+1} \leqslant 2 \frac{\alpha^i}{\sqrt{i+1}} = 2p_i.$$

For all j, we have

$$\sum_i \beta_{ij} p_i = \sum_{i=j+n}^{\infty} \frac{\alpha^{i-j}}{i+1} \frac{\alpha^i}{\sqrt{i+1}} = \sum_{i=j+n}^{\infty} \frac{\alpha^{2i-j}}{(i+1)^{3/2}}$$

$$\leqslant \alpha^{j+2n} \int_{j+n}^{\infty} \frac{dx}{x^{3/2}} = \frac{2\alpha^{j+2n}}{\sqrt{j+n}} \leqslant 2\alpha^{2n} \frac{\alpha^j}{\sqrt{j+1}} = 2\alpha^{2n} p_j.$$

It follows that $\| B_\alpha(n) \|^2 \leqslant 2(2\alpha^{2n}) = (2\alpha^n)^2$.

THEOREM 1. *A_λ is bounded for $|\lambda| \leqslant 1$ and* $\| A_\lambda \| \leqslant 1 + 2|\lambda|$.

PROOF. We observe that $D \equiv A_{|\lambda|} - B_{|\lambda|}(1)$ is the diagonal operator with diagonal $\{\frac{1}{n}\}_{n=1}^{\infty}$. Since $\| A_\lambda \| \leqslant \| A_{|\lambda|} \|$, it follows from Lemma 1 that

$$\|A_\lambda\| \leqslant \|B_{|\lambda|}(1) + D\| \leqslant 2|\lambda| + 1.$$

PROPOSITION. *A_λ is unitarily equivalent to $A_{|\lambda|}$*, $0 < |\lambda| \leqslant 1$.

PROOF. Consider the diagonal operator D_λ with diagonal $\{(\lambda/|\lambda|)^n\}_{n=0}^{\infty}$. It is routine to check that D_λ is unitary and $A_{|\lambda|} D_\lambda = D_\lambda A_\lambda$.

This result reduces our study to A_α, $0 \leqslant \alpha \leqslant 1$. The diagonal operator A_0 is compact and Hermitian and has spectrum $\{\frac{1}{n}\}_{n=1}^{\infty} \cup \{0\}$. The Cesàro operator A_1 was studied in **[1]** and **[4]**. We shall now restrict ourselves to $0 < \alpha < 1$.

THEOREM 2. *The point spectrum of A_α, $0 < \alpha < 1$, is the set $\{\frac{1}{n}\}_{n=1}^{\infty}$. The eigenvector corresponding to the simple eigenvalue $\frac{1}{n}$ has closed form $f(z) = z^{n-1}(1 - \alpha z)^{-n}$. The eigenvectors for A_α span H^2.*

PROOF. If $A_\alpha f = g$, then $f(0) = g(0)$, and if $n \geqslant 1$, then $f(n) = (n+1)g(n) - \alpha n g(n-1)$. Consequently, if $A_\alpha f = \gamma f$, then $f(n) = \gamma((n+1)f(n) - \alpha n f(n-1))$, or $(\gamma(n+1) - 1)f(n) = \alpha n \gamma f(n-1)$ for $n \geqslant 1$. If m is the smallest integer for which $f(m) \neq 0$, then $\gamma = 1/(m+1)$, so $0 < \gamma \leqslant 1$. Thus $f(n) = 0$ for $n < m$ and $f(n) = \alpha n f(n-1)/(n-m)$ for $n > m$. This implies that

$$f(m+n) = \alpha^n \prod_{j=1}^{n} \left(\frac{m}{j} + 1 \right) f(m), \qquad n \geqslant 1. \tag{1}$$

We conclude from this that all the eigenvalues are simple. Since

$$\frac{|f(m+n+1)|^2}{|f(m+n)|^2} = \alpha^2 \left(\frac{m}{n+1} + 1 \right)^2 \to \alpha^2 \quad \text{as } n \to \infty,$$

the ratio test implies that $f \in l^2$. Therefore $1/(m+1)$ is a simple eigenvalue for A; from (1) we find that the corresponding eigenvector has the form

$$f(z) = z^m + \sum_{n=1}^{\infty} \alpha^n \frac{(m+1)(m+2) \cdots (m+n)}{n!} z^{m+n}$$
$$= z^m (1 - \alpha z)^{-(m+1)}.$$

Finally, assume $g \in H^2$ and g is orthogonal to all the eigenvectors of A_α. Then

$$\frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})\,\overline{(e^{i\theta})^m(1-\alpha e^{i\theta})^{-m-1}}\,d\theta = 0 \quad \text{for all } m,$$

so

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta} g(e^{i\theta})\,\overline{\psi_\alpha(e^{i\theta})^{m+1}}\,d\theta = 0$$

for all m, where $\psi_\alpha(z) = z(1-\alpha z)^{-1}$. Since ψ_α is analytic in $D \equiv \{z \colon |z| < 1\}$ and is continuous and univalent in $\overline{D}$, it follows that for any nonnegative integer k there exists a sequence of polynomials $\{p_j\}$ such that $p_j \circ \psi_\alpha \to z^k$ uniformly on $\overline{D}$; as a consequence, $\{\psi_\alpha^n\}_{n=0}^\infty$ spans H^2 [**5**, p. 8]. Therefore $zg(z)$ is constant in H^2. Hence $g = 0$. This completes the proof.

Before proceeding, we remark that in case $\alpha < 1$, A_α can be shown to have the following closed form:

$$(A_\alpha f)(z) = z^{-1}\int_0^z (1-\alpha s)^{-1} f(s)\,ds, \qquad |z| < 1.$$

THEOREM 3. *The point spectrum of A_α^*, $0 < \alpha < 1$, is the set $\{\frac{1}{n}\}_{n=1}^\infty$. The eigenvector corresponding to the simple eigenvalue $\frac{1}{n}$ has closed form $f(z) = (\alpha - z)^{n-1}$. The eigenvectors for A_α^* span H^2.*

PROOF. Observe first that $(A_\alpha^* f)(n) = \sum_{k=n}^\infty \alpha^{k-n} f(k)/(k+1)$. If $A_\alpha^* f = g$, then $f(n) = (n+1)(g(n) - \alpha g(n+1))$ for $n = 0, 1, 2, \ldots$. Consequently, if $A_\alpha^* f = \gamma f$, then $f(n) = \gamma(n+1)(f(n) - \alpha f(n+1))$. It follows that 0 is not an eigenvalue of A_α^* (if $\gamma = 0$, then $f(n) = 0$ for all n). Therefore $f(n+1) = \alpha^{-1}[1 - 1/\gamma(n+1)]f(n)$. This implies that if $n \geqslant 1$, then

$$f(n) = \alpha^{-n}\prod_{j=1}^n \left[1 - \frac{1}{j\gamma}\right] f(0). \tag{2}$$

From this we conclude that all the eigenvalues are simple. Assume $\gamma \notin \{\frac{1}{m}\}_{m=1}^\infty$. Then $|f(n+1)|^2/|f(n)|^2 \to \alpha^{-2} > 1$ as $n \to \infty$, and it follows from the ratio test that $f \notin l^2$. Now assume $\gamma = \frac{1}{m}$, m a positive integer. Take $f(0) = \alpha^{m-1}$ in (2); then

$$f(z) = \sum_{k=0}^{m-1}\binom{m-1}{k}\alpha^{m-1-k}(-z)^k = (\alpha - z)^{m-1},$$

so $f \in H^2$. Hence $\frac{1}{m}$ is an eigenvalue for A_α^*. The eigenvector corresponding to eigenvalue $\frac{1}{m}$ is a polynomial of degree $m-1$; this makes it clear that the eigenvectors for A_α^* span H^2.

THEOREM 4. *For $0 < \alpha < 1$, A_α is compact and $\sigma(A_\alpha)$ (the spectrum of A_α) is the set $\{\frac{1}{n}\}_{n=1}^\infty \cup \{0\}$. A_α is not hyponormal (and hence not subnormal) if $0 < \alpha < 1$.*

PROOF. Observe that

$$\|A_\alpha - (A_\alpha - B_\alpha(n))\| = \|B_\alpha(n)\| \leqslant 2\alpha^n$$

for n a positive integer by Lemma 1. Letting $n \to \infty$ we see that A_α is the norm limit of the sequence of compact operators $A_\alpha - B_\alpha(n)$ and is therefore compact. Since

$\sigma(A_\alpha)$ is closed and $\{\frac{1}{n}\}_{n=1}^{\infty} \subseteq \sigma(A_\alpha)$ (by Theorem 2), we must have $0 \in \sigma(A_\alpha)$. Since A_α is compact we know that if $\gamma \neq 0$ and $\gamma = \sigma(A_\alpha)$ then $\gamma \in \pi_0(A_\alpha)$ (the point spectrum of A_α) and $\gamma \in \pi_0(A_\alpha^*)$ [**6**, p. 103]. It follows that $\sigma(A_\alpha) = \{\frac{1}{n}\}_{n=1}^{\infty} \cup \{0\}$. If A_α were hyponormal, it would be true that $\|A_\alpha\| = r(A_\alpha)$ (the spectral radius of A_α) [**3**, Problem 162]. We just determined that $r(A_\alpha) = 1$ if $\alpha < 1$. Since $\|A_\alpha\| \geqslant \|A_\alpha 1\| > 1$, it is clear that A_α cannot be hyponormal if $0 < \alpha < 1$.

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