PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 3, November 1982

THE TRANSLATION INVARIANT UNIFORM APPROXIMATION PROPERTY FOR COMPACT GROUPS

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ABSTRACT. We show that, unlike the abelian case, the translation invariant uniform approximation property fails in a strong way for L^1 of a compact connected semisimple Lie group.

1. We recall (see [**5**, **4** and **1**]) that a Banach space X is said to have the uniform approximation property if there are $k \geqslant 1$ and a sequence $q(n)$ of positive numbers such that for any m-dimensional subspace $E \subset X$ there exists an operator $T: X \to X$ for which $Tx = x$ for $x \in E$, $\|T\| \leqslant k$, $\dim TX \leqslant q(m)$.

As an important example we recall (see [**4**]) that the reflexive Orlicz spaces have the uniform approximation property. The purpose of this paper is to show that the situation is strongly different when we consider a Banach function space on a compact group and the translation invariant analogue of the uniform approximation property, i.e. if we modify the above definition assuming that X, E and T are translation invariant. This notion was introduced by Bożeiko and Pełczński in [**1**], where the case of a compact abelian group was considered; they proved the theorem below and they obtained, as a consequence, a general result on the translation invariant analogue of the uniform approximation property for translation invariant function Banach spaces on G.

THEOREM 1 (BOŻEIKO AND PEŁCZIŃSKI). *Let G be a compact abelian group with dual group Γ. For every $k > 1$ there exists a positive sequence $q_k(n)$ such that for every finite set $M \subset \Gamma$ there exists a trigonometric polynomial g such that $\hat{g}|_M = 1$, $\|g\|_1 \leqslant k$,* $\operatorname{card}(\operatorname{supp}(\hat{g})) \leqslant q_k(\operatorname{card} M)$.

The above theorem is no longer true for arbitrary compact groups. We shall prove the following

THEOREM 2. *Let G be a compact connected semisimple Lie group with dual object $\hat{G}$ (a maximal set of pairwise inequivalent unitary irreducible representations of G). Let $m \geqslant 1$, let $\{M_h\}_{h=1}^{\infty}$ be a sequence of pairwise disjoint subsets of $\hat{G}$ of cardinality at most m and let $\{P_h\}_{h=1}^{\infty}$ be a sequence of trigonometric polynomials such that $\hat{P}_h(\sigma) = I_\sigma$ for all $\sigma \in M_h$ and* $\operatorname{card}(\operatorname{supp}(\hat{P}_h)) \leqslant \operatorname{const}_m$, *then $\|P_h\|_1 \to \infty$ (as $h \to \infty$).*

The author wishes to thank P. M. Soardi for raising the question considered here.
In the next section we shall fix the notation.

Received by the editors August 4, 1981.
1980 *Mathematics Subject Classification*. Primary 43A15; Secondary 22E46.

0002-9939/82/0000-0223/$01.75

2. Let G denote a compact connected semisimple Lie group with Lie algebra $\mathfrak{g}$ and T a maximal torus with Lie algebra $\mathfrak{t}$. The complexification $\mathfrak{t}_{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$ and we denote by Δ the set of roots of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$. We choose in Δ a system P of positive roots and the associated system of simple roots. We set $\beta = \frac{1}{2}\Sigma_{\alpha \in P}\alpha$. The weights of G are ordered by letting $\lambda_1 \preccurlyeq \lambda_2$ if $\lambda_2 - \lambda_1$ is a sum (possibly empty) of simple roots [**6**, p. 314]. We write $\lambda_1 \prec \lambda_2$ if $\lambda_1 \preccurlyeq \lambda_2$ and $\lambda_1 \neq \lambda_2$. If E is a set of weights of G, we say that λ is a minimal (maximal) weight in E if no weight γ in E satisfies $\gamma \prec \lambda$ ($\gamma \succ \lambda$). For every weight λ the symmetric sum $S(\lambda)$ is defined on T by $S(\lambda)(t) = \Sigma_\mu \exp \mu(u)$, where $t = \exp u$ ($u \in \mathfrak{t}$) and the summation runs over all μ in the orbit of λ under the action of the Weyl group W; we denote by $\tilde{S}(\lambda)$ the unique central continuous extension of $S(\lambda)$ to the whole of G. We recall that the dual object $\hat{G}$ may be identified with the semilattice Σ of the dominant weights of G; throughout this paper we shall use the same symbol to denote a representation in $\hat{G}$ and his highest weight in Σ. Finally, for every λ in $\hat{G}$, χ_λ and d_λ denote the character and the dimension of λ respectively, while I_λ is the identity operator on the Hilbert space H_{d_λ}.

3. We need a lemma, whose proof is contained in [**2**].

LEMMA. *For any λ in Σ the function $\tilde{S}(\lambda + 2\beta)$ has the following Fourier expansion (on G): $\tilde{S}(\lambda + 2\beta) = \chi_{\lambda+2\beta} + \Sigma m_\gamma \chi_\gamma + \tau\chi_\lambda$, where $\tau = \pm 1$, the m_γ are relative integers and $\lambda \prec \gamma \prec \lambda + 2\beta$ for any γ in the above summation.*

PROOF OF THEOREM 2. We choose a maximal weight λ_h in M_h and we write $F_1 = \operatorname{supp}(\hat{\tilde{S}}(\lambda_h + 2\beta)) \cap \operatorname{supp}(\hat{P}_h)$; we have two cases:

(a^1) $F_1 = \{\lambda_h\}$,

(b^1) $F_1 = \{\lambda_h, \gamma_1^1, \ldots, \gamma_r^1\}$ ($r \geqslant 1$).

In the first case we obtain

$$\|P_h\|_1 \geqslant \frac{1}{\|\tilde{S}(\lambda_h + 2\beta)\|_\infty} \cdot \|P_h * \tilde{S}(\lambda_h + 2\beta)\|_\infty$$
$$\geqslant \frac{1}{\operatorname{card} W} \cdot \|\chi_{\lambda_h}\|_\infty \to \infty \quad (\text{as } h \to \infty).$$

In the case (b^1) we first observe that (by the lemma and the maximality of λ_h) we have $M_h \cap \{\gamma_1^1, \ldots, \gamma_r^1\} = \varnothing$. Then we choose a minimal weight $\bar{\gamma}_1^1$ in $\{\gamma_1^1, \ldots, \gamma_r^1\}$ and an integer n_1 such that the polynomial $T_1^1 = \tau\tilde{S}(\lambda_h + 2\beta) + n_1\tilde{S}(\gamma_1^1 + 2\beta)$ satisfies $\hat{T}_1^1(\gamma_1^1) = 0$; observe that $\hat{T}_1^1(\lambda_h) = d_{\lambda_h}^{-1} I_{\lambda_h}$, otherwise, by the lemma, we should have $\lambda_h \prec \bar{\gamma}_1^1$ and $\bar{\gamma}_1^1 \prec \lambda_h$; in the same way one verifies that $\hat{T}_1^1(\lambda) = 0$ for any $\lambda \neq \lambda_h$ in M_h. Then we choose a minimal weight $\bar{\gamma}_2^1$ in $\{\gamma_1^1, \ldots, \gamma_r^1\} \setminus \{\bar{\gamma}_1^1\}$ and an integer n_2 such that $T_2^1 = T_1^1 + n_2\tilde{S}(\bar{\gamma}_2^1 + 2\beta)$ satisfies $\hat{T}_2^1(\bar{\gamma}_2^1) = 0$; observe that $\hat{T}_2^1(\lambda_h) = d_{\lambda_h}^{-1} I_{\lambda_h}$, while $T_2^1(\bar{\gamma}_1^1) = 0$ (we cannot have $\bar{\gamma}_2^1 \preccurlyeq \bar{\gamma}_1^1$, hence, by the lemma,

$$\bar{\gamma}_1^1 \notin \operatorname{supp}\left(\hat{\tilde{S}}\left(\bar{\gamma}_2^1 + 2\beta\right)\right));$$

moreover, as above, $\hat{T}_2^1(\lambda) = 0$ for any $\lambda \neq \lambda_h$ in M_h. Now we choose a minimal weight $\bar{\gamma}_3^1$ in $\{\bar{\gamma}_1^1, \ldots, \bar{\gamma}_r^1\} \setminus \{\bar{\gamma}_1^1, \bar{\gamma}_2^1\}$ and an integer n_3 such that $T_3^1 = T_2^1 + n_3\tilde{S}(\bar{\gamma}_3^1 + 2\beta)$ satisfies $\hat{T}_3^1(\bar{\gamma}_3^1) = 0$; arguing as above we observe that $\hat{T}_3^1(\lambda_h) = d_{\lambda_h}^{-1} I_{\lambda_h}$, $\hat{T}_3^1(\gamma_i) = 0$ ($i = 1, 2$), $\hat{T}_3^1(\lambda) = 0$ for any $\lambda \neq \lambda_h$ in M_h. We go on until we construct a trigonometric polynomial $T^1 = T_r^1$ with the following properties: $\hat{T}^1(\lambda_h) = d_{\lambda_h}^{-1} I_{\lambda_h}$; $\hat{T}^1(\gamma_j) = 0$ for all $j = 1, \ldots, r$; $\hat{T}^1(\lambda) = 0$ for any $\lambda \neq \lambda_h$ in M_h; $\|T^1\|_\infty \leqslant \text{const}_W$ (in particular $\|T^1\|_\infty$ does not depend upon h).

Now we write $F_2 = \text{supp}(\hat{T}^1) \cap \text{supp}(\hat{P}_h)$; we have two cases:

(a^2) $F_2 = \{\lambda_h\}$,

(b^2) $F_2 = \{\lambda_h, \gamma_1^2, \ldots, \gamma_s^2\}$ $(s \geqslant 1)$

(observe that, by construction, $\{\gamma_1^1, \ldots, \gamma_r^1\} \cap \{\gamma_1^2, \ldots, \gamma_s^2\} = \varnothing$; observe also that for any γ_j^2 we have $\lambda_h \prec \gamma_j^2$, hence we cannot have $\gamma_j^2 \preccurlyeq \lambda$ for any λ in M_h). In the case (a^2) we obtain, as above,

$$\|P_h\|_1 \geqslant \frac{1}{\|T^1\|_\infty} \cdot \|P_h * T^1\|_\infty \geqslant \text{const}_W^{-1} \cdot d_{\lambda_h} \to \infty \quad (\text{as } h \to \infty).$$

In the case (b^2) we argue exactly as above (taking the set $\{\gamma_1^1, \ldots, \gamma_r^1, \gamma_1^2, \ldots, \gamma_s^2\}$ in place of $\{\gamma_1^1, \ldots, \gamma_r^1\}$) until we construct a trigonometric polynomial T^2 which has the following properties: $\hat{T}^2(\lambda_h) = d_{\lambda_h}^{-1} I_{\lambda_h}$; $\hat{T}^2(\gamma_j^1) = 0$ for all $j = 1, \ldots, r$; $\hat{T}^2(\gamma_j^2) = 0$ for all $j = 1, \ldots, s$; $\|T^2\|_\infty \leqslant \text{const}_W$.

Then we write $F_3 = \text{supp}(\hat{T}^2) \cap \text{supp}(\hat{P}_h)$; again we have two cases (a^3) and (b^3) and we go on. To complete the proof we recall that $\text{card}(\text{supp}(\hat{P}_h)) \leqslant \text{const}_m$, hence we cannot be in the second case for more than const_m steps.

REMARK. In **[3]** it was shown that $\|\chi_\sigma\|_3 \to \infty$ as σ runs in Σ. Hence the technique of the above proof shows that not only $\|P_h\|_1 \to \infty$, but also the (L^p, L^p) convolutor norm of P_h diverges for $1 \leqslant p \leqslant \frac{3}{2}$ or $p \geqslant 3$. We omit the details.

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