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THE TRANSLATION INVARIANT UNIFORM APPROXIMATION PROPERTY FOR COMPACT GROUPS

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ABSTRACT. We show that, unlike the abelian case, the translation invariant uniform approximation property fails in a strong way for L^1 of a compact connected semisimple Lie group.

1. We recall (see [**5**, **4** and **1**]) that a Banach space X is said to have the uniform approximation property if there are $k \geqslant 1$ and a sequence $q(n)$ of positive numbers such that for any m-dimensional subspace $E \subset X$ there exists an operator $T: X \to X$ for which $Tx = x$ for $x \in E$, $\|T\| \leqslant k$, $\dim TX \leqslant q(m)$.

As an important example we recall (see [**4**]) that the reflexive Orlicz spaces have the uniform approximation property. The purpose of this paper is to show that the situation is strongly different when we consider a Banach function space on a compact group and the translation invariant analogue of the uniform approximation property, i.e. if we modify the above definition assuming that X, E and T are translation invariant. This notion was introduced by Bożeiko and Pełcziński in [**1**], where the case of a compact abelian group was considered; they proved the theorem below and they obtained, as a consequence, a general result on the translation invariant analogue of the uniform approximation property for translation invariant function Banach spaces on G.

THEOREM 1 (BOŻEIKO AND PEŁCZIŃSKI). *Let G be a compact abelian group with dual group Γ. For every $k > 1$ there exists a positive sequence $q_k(n)$ such that for every finite set $M \subset \Gamma$ there exists a trigonometric polynomial g such that* $\hat{g}|_M = 1$, $\|g\|_1 \leqslant k$, $\operatorname{card}(\operatorname{supp}(\hat{g})) \leqslant q_k(\operatorname{card} M)$.

The above theorem is no longer true for arbitrary compact groups. We shall prove the following

THEOREM 2. *Let G be a compact connected semisimple Lie group with dual object $\hat{G}$ (a maximal set of pairwise inequivalent unitary irreducible representations of G). Let $m \geqslant 1$, let $\{M_h\}_{h=1}^{\infty}$ be a sequence of pairwise disjoint subsets of $\hat{G}$ of cardinality at most m and let $\{P_h\}_{h=1}^{\infty}$ be a sequence of trigonometric polynomials such that $\hat{P}_h(\sigma) = I_\sigma$ for all $\sigma \in M_h$ and* $\operatorname{card}(\operatorname{supp}(\hat{P}_h)) \leqslant \operatorname{const}_m$, *then $\|P_h\|_1 \to \infty$ (as $h \to \infty$).*

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In the next section we shall fix the notation.

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2. Let G denote a compact connected semisimple Lie group with Lie algebra $\mathfrak{g}$ and T a maximal torus with Lie algebra $\mathfrak{t}$. The complexification $\mathfrak{t}_{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$ and we denote by Δ the set of roots of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$. We choose in Δ a system P of positive roots and the associated system of simple roots. We set $\beta = \frac{1}{2}\Sigma_{\alpha \in P}\alpha$. The weights of G are ordered by letting $\lambda_1 \preccurlyeq \lambda_2$ if $\lambda_2 - \lambda_1$ is a sum (possibly empty) of simple roots [**6**, p. 314]. We write $\lambda_1 \prec \lambda_2$ if $\lambda_1 \preccurlyeq \lambda_2$ and $\lambda_1 \neq \lambda_2$. If E is a set of weights of G, we say that λ is a minimal (maximal) weight in E if no weight γ in E satisfies $\gamma \prec \lambda$ ($\gamma \succ \lambda$). For every weight λ the symmetric sum $S(\lambda)$ is defined on T by $S(\lambda)(t) = \Sigma_\mu \exp \mu(u)$, where $t = \exp u$ ($u \in \mathfrak{t}$) and the summation runs over all μ in the orbit of λ under the action of the Weyl group W; we denote by $\tilde{S}(\lambda)$ the unique central continuous extension of $S(\lambda)$ to the whole of G. We recall that the dual object $\hat{G}$ may be identified with the semilattice Σ of the dominant weights of G; throughout this paper we shall use the same symbol to denote a representation in $\hat{G}$ and his highest weight in Σ. Finally, for every λ in $\hat{G}$, χ_λ and d_λ denote the character and the dimension of λ respectively, while I_λ is the identity operator on the Hilbert space H_{d_λ}.

3. We need a lemma, whose proof is contained in [**2**].

LEMMA. *For any λ in Σ the function $\tilde{S}(\lambda + 2\beta)$ has the following Fourier expansion (on G): $\tilde{S}(\lambda + 2\beta) = \chi_{\lambda+2\beta} + \Sigma\, m_\gamma \chi_\gamma + \tau\chi_\lambda$, where $\tau = \pm 1$, the m_γ are relative integers and $\lambda \prec \gamma \prec \lambda + 2\beta$ for any γ in the above summation.*

PROOF OF THEOREM 2. We choose a maximal weight λ_h in M_h and we write $F_1 = \operatorname{supp}(\hat{\tilde{S}}(\lambda_h + 2\beta)) \cap \operatorname{supp}(\hat{P}_h)$; we have two cases:

(a^1) $F_1 = \{\lambda_h\}$,

(b^1) $F_1 = \{\lambda_h, \gamma_1^1, \ldots, \gamma_r^1\}$ ($r \geqslant 1$).

In the first case we obtain

$$\|P_h\|_1 \geqslant \frac{1}{\|\tilde{S}(\lambda_h + 2\beta)\|_\infty} \cdot \|P_h * \tilde{S}(\lambda_h + 2\beta)\|_\infty$$
$$\geqslant \frac{1}{\operatorname{card} W} \cdot \|\chi_{\lambda_h}\|_\infty \to \infty \quad (\text{as } h \to \infty).$$

In the case (b^1) we first observe that (by the lemma and the maximality of λ_h) we have $M_h \cap \{\gamma_1^1, \ldots, \gamma_r^1\} = \varnothing$. Then we choose a minimal weight $\bar{\gamma}_1^1$ in $\{\gamma_1^1, \ldots, \gamma_r^1\}$ and an integer n_1 such that the polynomial $T_1^1 = \tau\tilde{S}(\lambda_h + 2\beta) + n_1\tilde{S}(\gamma_1^1 + 2\beta)$ satisfies $\hat{T}_1^1(\gamma_1^1) = 0$; observe that $\hat{T}_1^1(\lambda_h) = d_{\lambda_h}^{-1}I_{\lambda_h}$, otherwise, by the lemma, we should have $\lambda_h \prec \bar{\gamma}_1^1$ and $\bar{\gamma}_1^1 \prec \lambda_h$; in the same way one verifies that $\hat{T}_1^1(\lambda) = 0$ for any $\lambda \neq \lambda_h$ in M_h. Then we choose a minimal weight $\bar{\gamma}_2^1$ in $\{\gamma_1^1, \ldots, \gamma_r^1\} \setminus \{\bar{\gamma}_1^1\}$ and an integer n_2 such that $T_2^1 = T_1^1 + n_2\tilde{S}(\bar{\gamma}_2^1 + 2\beta)$ satisfies $\hat{T}_2^1(\bar{\gamma}_2^1) = 0$; observe that $\hat{T}_2^1(\lambda_h) = d_{\lambda_h}^{-1}I_{\lambda_h}$, while $T_2^1(\bar{\gamma}_1^1) = 0$ (we cannot have $\bar{\gamma}_2^1 \preccurlyeq \bar{\gamma}_1^1$, hence, by the lemma,

$$\bar{\gamma}_1^1 \notin \operatorname{supp}\left(\hat{\tilde{S}}\left(\bar{\gamma}_2^1 + 2\beta\right)\right));$$

moreover, as above, $\hat{T}_2^1(\lambda) = 0$ for any $\lambda \neq \lambda_h$ in M_h. Now we choose a minimal weight $\bar{\gamma}_3^1$ in $\{\bar{\gamma}_1^1, \ldots, \bar{\gamma}_r^1\} \setminus \{\bar{\gamma}_1^1, \bar{\gamma}_2^1\}$ and an integer n_3 such that $T_3^1 = T_2^1 + n_3\tilde{S}(\bar{\gamma}_3^1 + 2\beta)$ satisfies $\hat{T}_3^1(\bar{\gamma}_3^1) = 0$; arguing as above we observe that $\hat{T}_3^1(\lambda_h) = d_{\lambda_h}^{-1}I_{\lambda_h}$, $\hat{T}_3^1(\gamma_i) = 0$ ($i = 1, 2$), $\hat{T}_3^1(\lambda) = 0$ for any $\lambda \neq \lambda_h$ in M_h. We go on until we construct a trigonometric polynomial $T^1 = T_r^1$ with the following properties: $\hat{T}^1(\lambda_h) = d_{\lambda_h}^{-1} I_{\lambda_h}$; $\hat{T}^1(\gamma_j) = 0$ for all $j = 1, \ldots, r$; $\hat{T}^1(\lambda) = 0$ for any $\lambda \neq \lambda_h$ in M_h; $\|T^1\|_\infty \leqslant \text{const}_W$ (in particular $\|T^1\|_\infty$ does not depend upon h).

Now we write $F_2 = \text{supp}(\hat{T}^1) \cap \text{supp}(\hat{P}_h)$; we have two cases:

(a^2) $F_2 = \{\lambda_h\}$,

(b^2) $F_2 = \{\lambda_h, \gamma_1^2, \ldots, \gamma_s^2\}$ $(s \geqslant 1)$

(observe that, by construction, $\{\gamma_1^1, \ldots, \gamma_r^1\} \cap \{\gamma_1^2, \ldots, \gamma_s^2\} = \varnothing$; observe also that for any γ_j^2 we have $\lambda_h \prec \gamma_j^2$, hence we cannot have $\gamma_j^2 \preccurlyeq \lambda$ for any λ in M_h). In the case (a^2) we obtain, as above,

$$\|P_h\|_1 \geqslant \frac{1}{\|T^1\|_\infty} \cdot \|P_h * T^1\|_\infty \geqslant \text{const}_W^{-1} \cdot d_{\lambda_h} \to \infty \quad (\text{as } h \to \infty).$$

In the case (b^2) we argue exactly as above (taking the set $\{\gamma_1^1, \ldots, \gamma_r^1, \gamma_1^2, \ldots, \gamma_s^2\}$ in place of $\{\gamma_1^1, \ldots, \gamma_r^1\}$) until we construct a trigonometric polynomial T^2 which has the following properties: $\hat{T}^2(\lambda_h) = d_{\lambda_h}^{-1} I_{\lambda_h}$; $\hat{T}^2(\gamma_j^1) = 0$ for all $j = 1, \ldots, r$; $\hat{T}^2(\gamma_j^2) = 0$ for all $j = 1, \ldots, s$; $\|T^2\|_\infty \leqslant \text{const}_W$.

Then we write $F_3 = \text{supp}(\hat{T}^2) \cap \text{supp}(\hat{P}_h)$; again we have two cases (a^3) and (b^3) and we go on. To complete the proof we recall that $\text{card}(\text{supp}(\hat{P}_h)) \leqslant \text{const}_m$, hence we cannot be in the second case for more than const_m steps.

REMARK. In [3] it was shown that $\|\chi_\sigma\|_3 \to \infty$ as σ runs in Σ. Hence the technique of the above proof shows that not only $\|P_h\|_1 \to \infty$, but also the (L^p, L^p) convolutor norm of P_h diverges for $1 \leqslant p \leqslant \frac{3}{2}$ or $p \geqslant 3$. We omit the details.

REFERENCES

1. M. Bożeiko and A. Pełczyński, *An analogue in commutative harmonic analysis of the uniform approximation property of Banach spaces*, Séminaire d'Analyse Fonctionnelle (1978–1979), Exp. No. 9, École Polytech. Palaiseau, 1979.

2. S. Giulini, P. M. Soardi and G. Travaglini, *A Cohen type inequality for compact Lie groups*, Proc. Amer. Math. Soc. **77** (1979), 359–364.

3. ______, *Norms of characters and Fourier series on compact Lie groups*, J. Funct. Anal. **46** (1982), 88–101.

4. J. Linderstrauss and L. Tzafriri, *The uniform approximation property in Orlicz spaces*, Israel J. Math. **23** (1976), 142–155.

5. A. Pełczyński and H. P. Rosenthal, *Localization techniques in L_p spaces*, Studia Math. **52** (1975), 263–289.

6. V. S. Varadarajan, *Lie groups, Lie algebras and their representations*, Prentice-Hall, Englewood Cliffs, N. J., 1974.

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