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THE DIMENSION OF PEAK-INTERPOLATION SETS

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ABSTRACT. The dimension of a peak-interpolation set in the boundary of a strongly pseudoconvex domain in $\mathbf{C}^N$ does not exceed $N-1$.

Recall that a *peak set* for a domain D in $\mathbf{C}^N$ is a subset E of bD for which there is $f \in A(D)$, i.e., an f continuous on $\bar{D}$, holomorphic on D, with $f = 1$ on E and $|f| < 1$ on $\bar{D} \setminus E$. The set E is a *peak-interpolation set* for D if given a nonzero $\varphi \in \mathcal{C}(E)$, there is $f \in A(D)$ with $f = \varphi$ on E, $|f| < \sup\{|\varphi(x)|: x \in E\}$ on $\bar{D} \setminus E$. For strongly pseudoconvex domains, a peak set is a peak-interpolation set. The general theory of these sets is given in [**12**].

Very little is known about the structure of peak-interpolation sets for domains in $\mathbf{C}^N$, but W. Rudin conjectured [**11**] that a peak-interpolation set for a strongly pseudoconvex domain in $\mathbf{C}^N$ has dimension not more than $N-1$. In [**5**] it was shown that this dimension cannot exceed N. The present paper is devoted to a proof of Rudin's conjecture: The correct bound is $N-1$. We obtain this result for a class of domains more extensive than the class of strongly pseudoconvex domains.

The idea of the proof, in $\mathbf{C}^2$, is that, as shown by Frankl and Pontrjagin [**6**] (cf. [**7** and **2**]), a closed, 2-dimensional subset of $\mathbf{R}^3$ disconnects some open connected subset of $\mathbf{R}^3$. (Curiously, this result seems not to have found a place in the modern texts on dimension theory.) In the case of domains in $\mathbf{C}^N$, we use a suitable cohomological generalization of this fact given in [**10**].

We should note explicitly that we understand *dimension* in the topological sense; the example of Tumanov [**14**] shows that there is no such result for metric dimension. [ADDED IN PROOF. A more definitive example has been obtained by B. S. Henriksen. She has constructed a peak-interpolation set of Hausdorff dimension $2N-1$ in the boundary of a strongly pseudoconvex domain in $\mathbf{C}^N$. See Math. Ann. **259** (1982), 271–277.]

To formulate the result we prove, let us recall that a point p in the boundary of a convex domain Δ is said to be *rstrongly exposed* if there are neighborhoods of p in $b\Delta$ of arbitrarily small diameter and of the form $\{z \in bD: L(z) < 0\}$ where L is a real-valued, real affine functional on $\mathbf{C}^N$ with $L(p) = 0$. Equivalently, it is possible to cut off from $\bar{\Delta}$ arbitrarily small neighborhoods of p in $\bar{\Delta}$ with real hyperplanes. Each point in the boundary of the ball has this property as does each point in the distinguished boundary of the polydisc.

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THEOREM. *Let* Δ *be a bounded open convex set in* $\mathbf{C}^N$. *If* $F \subset b\Delta$ *is a peak set that consists entirely of strongly exposed points, then* $\dim F \leqslant N - 1$.

This result implies the corresponding result for smoothly bounded, strongly pseudoconvex domains, for the question is local, and near each point in the boundary of a smoothly bounded strongly pseudoconvex domain D, bD is strictly convex with respect to some set of local holomorphic coordinates.

PROOF OF THEOREM. Notice first that the set F is necessarily polynomially convex. Fix a real hyperplane Π in $\mathbf{C}^N$ that meets Δ, and let F^+ denote the intersection of F with one of the closed halfspaces determined by Π. The set $S = (\Pi \cap \bar{\Delta}) \cup F^+$ is easily seen to be polynomially convex: If S differs from its polynomially convex hull, $\hat{S}$, choose $p \in \hat{S} \setminus S$, and let μ be a Jensen measure for p supported on S. (For Jensen measures, see [**4** or **13**].) As F is a peak set for $A(\Delta)$, there is $g \in A(\Delta)$ that vanishes identically on F and that satisfies $g(p) = 1$. From

$$0 = \log|g(p)| \leqslant \mu(\log|g|)$$

we find that μ is concentrated on $\Pi \cap \bar{\Delta}$ whence, by convexity, $p \in \Pi \cap \bar{\Delta} \subset S$, a contradiction.

For the sake of clarity, we treat the case $N = 2$ of our theorem separately from the case of general N. This not necessary on logical grounds, but it may shed some light on the result.

Assume then that $N = 2$ and $\dim F = 2$. By convexity, bD is topologically equivalent to the three-sphere S^3. According to the result of Frankl and Pontrjagin cited above, there is a connected open set U in bD such that $U \setminus F$ is not connected. It follows, from the assumption that the points of E are strongly exposed, that for some $p \in F$, there is a real hyperplane Π that meets Δ and that passes so near p that if W is the component of $b\Delta \setminus \Pi$ containing p, then $W \setminus F$ is not connected. Thus, the compact, polynomially convex set $(\Pi \cap \bar{\Delta}) \cup (W \cap F)$ disconnects the topological three-sphere $(\Pi \cap \bar{\Delta}) \cup W = \Sigma$. The surface Σ is the boundary of a convex domain, viz., one of the components, Δ^+, of $\Delta \setminus \Pi$. According to a result of Alexander [**1**], no polynomially convex subset of $S^{2N-1} = bB_N$, B_N the unit ball in $\mathbf{C}^N$, disconnects S^{2N-1}. Alexander's proof applies verbatim when B_N is replaced by an arbitrary bounded, open convex domain. Applied to $b\Delta^+ = (\Pi \cap \bar{\Delta}) \cup W$ and the polynomially convex set $(\Pi \cap \bar{\Delta}) \cup (W \cap F)$, we see that we have a contradiction. This establishes the case $N = 2$ of our Theorem.

We now take up the case of general N. Thus, assume $\Delta \subset \mathbf{C}^N$ and assume, for the sake of contradiction, that $\dim F = N$. According to Theorem 2 of [**10**, p. 7], there is a point $p \in F$ that has a neighborhood U_0 in $b\Delta$ with the property that for every open V with $p \in V \subset U_0$, the natural map

$$\iota_{VU_0}\colon H_*^N(V \cap F) \to H_*^N(U_0 \cap F)$$

is nonzero. (All of our cohomology groups have coefficients in the integers, but we shall surpress the coefficient group from the notation. The star denotes cohomology with compact supports.)

Fix a real hyperplane Π passing through Δ and missing p, Π so close to p that the component W of $b\Delta \setminus \Pi$ containing p is contained in U_0. The set $F^\dagger = (F \cap \overline{W}) \cup (\Pi \cap \overline{\Delta})$ is polynomially convex, and so

$$H^k(F^\dagger) = 0, \qquad k = N, N+1, \ldots.$$

(This is essentially due to Andreotti and Narasimhan [**3**]; for the rather formal deduction of the vanishing of these groups from what is written in [**3**], see [**5**].) The set $(\Pi \cap \overline{\Delta}) \cup W = \Sigma$ is homeomorphic to the sphere S^{2N-1}; it is a convex surface. From the exact cohomology sequence [**8**, p. 190]

$$\cdots \to H^k(\Sigma) \to H^k(F^\dagger) \to H_*^{k+1}(\Sigma \setminus F^\dagger) \to H^{k+1}(\Sigma) \to \cdots$$

and the fact that $H^r(\Sigma) = 0$ for $0 < r < 2N - 1$, we find

$$H_*^{k+1}(\Sigma \setminus F^\dagger) \simeq H^k(F^\dagger)$$

for $1 \leqslant k \leqslant 2n - 2$, and this yields

$$(1) \qquad H_*^{k+1}(\Sigma \setminus F^\dagger) = 0, \qquad k = N, N+1, \ldots, 2N-2.$$

On the other hand, we have, by the choice of W, that

$$H_*^N(W \cap F^\dagger) = H_*^N(W \cap F) \neq 0.$$

Consider the exact cohomology sequence

$$\cdots \to H_*^N(W) \to H_*^N(W \cap F^\dagger) \to H_*^{N+1}(W \setminus F^\dagger) \to H_*^{N+1}(W) \to \cdots.$$

As W is homeomorphic to $\mathbf{R}^{2N-1}$, we have $H_*^N(W) = 0$ whence the nonzero group $H_*^N(W \cap F^\dagger)$ injects into the group $H_*^{N+1}(W \setminus F^\dagger)$. Since the sets $\Sigma \setminus F^\dagger$ and $W \setminus F^\dagger$ coincide, we have reached a contradiction to (1). Thus, $\dim F < N$.

Notice that the proof just given does not require that the set F be a peak-interpolation set; it need only be a peak set. In the case of strongly pseudoconvex domains the two notions coincide, but it is not known that they do in the geometric setting of the theorem.

COROLLARY. *If Δ is a bounded, convex domain in $\mathbf{C}^N$, and if f is a nonconstant element of $A(\Delta)$ bounded by one in modulus, then the set $M = \{z \in b\Delta: |f(z)| = 1\}$ has dimension no more than N if it consists entirely of strongly exposed points.*

PROOF. By hypothesis, f maps $\overline{\Delta}$ to the closed unit disc $\overline{U}$ in $\mathbf{C}$, and $M = f^{-1}(bU)$. According to the Theorem, $\dim F^{-1}(z) \leqslant N - 1$ for all $z \in bU$, and so, as $\dim bU = 1$, $\dim M \leqslant N$ follows from [**9**, Theorem V1.7].

For strongly pseudoconvex domains, this result was given in [**5**].

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