THE DIMENSION OF PEAK-INTERPOLATION SETS

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Abstract. The dimension of a peak-interpolation set in the boundary of a strongly pseudoconvex domain in \( \mathbb{C}^N \) does not exceed \( N - 1 \).

Recall that a peak set for a domain \( D \) in \( \mathbb{C}^N \) is a subset \( E \) of \( bD \) for which there is \( f \in A(D) \), i.e., an \( f \) continuous on \( \overline{D} \), holomorphic on \( D \), with \( f = 1 \) on \( E \) and \(|f|<1\) on \( \overline{D} \setminus E \). The set \( E \) is a peak-interpolation set for \( D \) if given a nonzero \( \varphi \in \mathcal{C}(E) \), there is \( f \in A(D) \) with \( f = \varphi \) on \( E \), \(|f|<\sup\{|\varphi(x)|: x \in E\}\) on \( \overline{D} \setminus E \). For strongly pseudoconvex domains, a peak set is a peak-interpolation set. The general theory of these sets is given in [12].

Very little is known about the structure of peak-interpolation sets for domains in \( \mathbb{C}^N \), but W. Rudin conjectured [11] that a peak-interpolation set for a strongly pseudoconvex domain in \( \mathbb{C}^N \) has dimension not more than \( N - 1 \). In [5] it was shown that this dimension cannot exceed \( N \). The present paper is devoted to a proof of Rudin's conjecture: The correct bound is \( N - 1 \). We obtain this result for a class of domains more extensive than the class of strongly pseudoconvex domains.

The idea of the proof, in \( \mathbb{C}^2 \), is that, as shown by Frankl and Pontrjagin [6] (cf. [7 and 2]), a closed, 2-dimensional subset of \( \mathbb{R}^3 \) disconnects some open connected subset of \( \mathbb{R}^3 \). (Curiously, this result seems not to have found a place in the modern texts on dimension theory.) In the case of domains in \( \mathbb{C}^N \), we use a suitable cohomological generalization of this fact given in [10].

We should note explicitly that we understand dimension in the topological sense; the example of Tumanov [14] shows that there is no such result for metric dimension. [ADDED IN PROOF. A more definitive example has been obtained by B. S. Henriksen. She has constructed a peak-interpolation set of Hausdorff dimension \( 2N - 1 \) in the boundary of a strongly pseudoconvex domain in \( \mathbb{C}^N \). See Math. Ann. 259 (1982), 271–277.]

To formulate the result we prove, let us recall that a point \( p \) in the boundary of a convex domain \( \Delta \) is said to be rstrongly exposed if there are neighborhoods of \( p \) in \( b\Delta \) of arbitrarily small diameter and of the form \( \{z \in bD: L(z) < 0\} \) where \( L \) is a real-valued, real affine functional on \( \mathbb{C}^N \) with \( L(p) = 0 \). Equivalently, it is possible to cut off from \( \Delta \) arbitrarily small neighborhoods of \( p \) in \( \Delta \) with real hyperplanes. Each point in the boundary of the ball has this property as does each point in the distinguished boundary of the polydisc.
Theorem. Let $\Delta$ be a bounded open convex set in $\mathbb{C}^N$. If $F \subset b\Delta$ is a peak set that consists entirely of strongly exposed points, then $\dim F \leq N - 1$.

This result implies the corresponding result for smoothly bounded, strongly pseudoconvex domains, for the question is local, and near each point in the boundary of a smoothly bounded strongly pseudoconvex domain $D$, $bD$ is strictly convex with respect to some set of local holomorphic coordinates.

Proof of Theorem. Notice first that the set $F$ is necessarily polynomially convex. Fix a real hyperplane $\Pi$ in $\mathbb{C}^N$ that meets $\Delta$, and let $F^+$ denote the intersection of $F$ with one of the closed halfspaces determined by $\Pi$. The set $S = (\Pi \cap \bar{\Delta}) \cup F^+$ is easily seen to be polynomially convex: If $S$ differs from its polynomially convex hull, $\hat{S}$, choose $p \in \hat{S} \setminus S$, and let $\mu$ be a Jensen measure for $p$ supported on $S$. (For Jensen measures, see [4 or 13].) As $F$ is a peak set for $A(\Delta)$, there is $g \in A(\Delta)$ that vanishes identically on $F$ and that satisfies $g(p) = 1$. From

$$0 = \log|g(p)| < \mu(\log|g|)$$

we find that $\mu$ is concentrated on $\Pi \cap \bar{\Delta}$ whence, by convexity, $p \in \Pi \cap \bar{\Delta} \subset S$, a contradiction.

For the sake of clarity, we treat the case $N = 2$ of our theorem separately from the case of general $N$. This not necessary on logical grounds, but it may shed some light on the result.

Assume then that $N = 2$ and $\dim F = 2$. By convexity, $bD$ is topologically equivalent to the three-sphere $S^3$. According to the result of Frankl and Pontrjagin cited above, there is a connected open set $U$ in $bD$ such that $U \setminus F$ is not connected. It follows, from the assumption that the points of $E$ are strongly exposed, that for some $p \in F$, there is a real hyperplane $\Pi$ that meets $\Delta$ and that passes so near $p$ that if $W$ is the component of $b\Delta \setminus \Pi$ containing $p$, then $W \setminus F$ is not connected. Thus, the compact, polynomially convex set $(\Pi \cap \bar{\Delta}) \cup (W \cap F)$ disconnects the topological three-sphere $(\Pi \cap \bar{\Delta}) \cup W = \Sigma$. The surface $\Sigma$ is the boundary of a convex domain, viz., one of the components, $\Delta^+$, of $\Delta \setminus \Pi$. According to a result of Alexander [1], no polynomially convex subset of $S^{2N-1} = bB_N$, $B_N$ the unit ball in $\mathbb{C}^N$, disconnects $S^{2N-1}$. Alexander’s proof applies verbatim when $B_N$ is replaced by an arbitrary bounded, open convex domain. Applied to $b\Delta^+ = (\Pi \cap \bar{\Delta}) \cup W$ and the polynomially convex set $(\Pi \cap \bar{\Delta}) \cup (W \cap F)$, we see that we have a contradiction. This establishes the case $N = 2$ of our Theorem.

We now take up the case of general $N$. Thus, assume $\Delta \subset \mathbb{C}^N$ and assume, for the sake of contradiction, that $\dim F = N$. According to Theorem 2 of [10, p. 7], there is a point $p \in F$ that has a neighborhood $U_0$ in $b\Delta$ with the property that for every open $V$ with $p \in V \subset U_0$, the natural map

$$t_{V,U_0}: H^N_*(V \cap F) \to H^N_*(U_0 \cap F)$$

is nonzero. (All of our cohomology groups have coefficients in the integers, but we shall surpress the coefficient group from the notation. The star denotes cohomology with compact supports.)
Fix a real hyperplane $\Pi$ passing through $\Delta$ and missing $p$, $\Pi$ so close to $p$ that the component $W$ of $b\Delta \setminus \Pi$ containing $p$ is contained in $U_0$. The set $F^+ = (F \cap \bar{W}) \cup (\Pi \cap \Delta)$ is polynomially convex, and so

$$H^k(F^+) = 0, \quad k = N, N + 1, \ldots.$$ (This is essentially due to Andreotti and Narasimhan [3]; for the rather formal deduction of the vanishing of these groups from what is written in [3], see [5].) The set $(\Pi \cap \Delta) \cup W = \Sigma$ is homeomorphic to the sphere $S^{2N-1}$; it is a convex surface. From the exact cohomology sequence [8, p. 190]

$$\cdots \to H^k(\Sigma) \to H^k(F^+) \to H^{k+1}(\Sigma \setminus F^+) \to H^{k+1}(\Sigma) \to \cdots$$

and the fact that $H^r(\Sigma) = 0$ for $0 < r < 2N - 1$, we find

$$H^{k+1}_*(\Sigma \setminus F^+) \cong H^k(F^+)$$

for $1 \leq k \leq 2n - 2$, and this yields

$$(1) \quad H^{k+1}_*(\Sigma \setminus F^+) = 0, \quad k = N, N + 1, \ldots, 2N - 2.$$ 

On the other hand, we have, by the choice of $W$, that

$$H^N_*(W \cap F^+) = H^N_*(W \cap F) \neq 0.$$ 

Consider the exact cohomology sequence

$$\cdots \to H^N_*(W) \to H^N_*(W \cap F^+) \to H^{N+1}_*(W \setminus F^+) \to H^{N+1}_*(W) \to \cdots.$$ 

As $W$ is homeomorphic to $R^{2N-1}$, we have $H^N_*(W) = 0$ whence the nonzero group $H^N_*(W \cap F^+)$ injects into the group $H^{N+1}_*(W \setminus F^+)$. Since the sets $\Sigma \setminus F^+$ and $W \setminus F^+$ coincide, we have reached a contradiction to (1). Thus, dim $F < N$.

Notice that the proof just given does not require that the set $F$ be a peak-interpolation set; it need only be a peak set. In the case of strongly pseudoconvex domains the two notions coincide, but it is not known that they do in the geometric setting of the theorem.

**Corollary.** If $\Delta$ is a bounded, convex domain in $C^N$, and if $f$ is a nonconstant element of $A(\Delta)$ bounded by one in modulus, then the set $M = \{ z \in b\Delta : |f(z)| = 1 \}$ has dimension no more than $N$ if it consists entirely of strongly exposed points.

**Proof.** By hypothesis, $f$ maps $\Delta$ to the closed unit disc $\Omega$ in $C$, and $M = f^{-1}(b\Omega)$. According to the Theorem, dim $F^-(z) \leq N - 1$ for all $z \in bU$, and so, as dim $bU = 1$, dim $M \leq N$ follows from [9, Theorem V1.7].

For strongly pseudoconvex domains, this result was given in [5].

**References**


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