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SEMIREGULAR INVARIANT MEASURES ON ABELIAN GROUPS

ANDRZEJ PELC

ABSTRACT. A nonnegative countably additive, extended real-valued measure is called semiregular if every set of positive measure contains a set of positive finite measure. V. Kannan and S. R. Raju [3] stated the problem of whether every invariant semiregular measure defined on all subsets of a group is necessarily a multiple of the counting measure. We prove that the negative answer is equivalent to the existence of a real-valued measurable cardinal.

It is shown, moreover, that a counterexample can be found on every abelian group of real-valued measurable cardinality.

We consider countably additive, nonnegative extended real-valued measures which are not identically equal to zero. Such a measure is called universal on a set X if it is defined on all subsets of X and it is called semiregular if every set of positive measure contains a set of positive finite measure. A universal measure m on a group $(G, \circ)$ is invariant if $m(a \circ A) = m(A)$ for every $a \in G, A \subset G$.

A measure is called κ-additive if the union of less than κ sets of measure 0 has measure 0. A cardinal κ is called real-valued measurable if there exists a finite universal κ-additive measure on κ which vanishes on singletons.[1] It is well known that the existence of a real-valued measurable cardinal is unprovable in usual set theory with choice.

Erdös and Mauldin [2] proved that if $(G, \circ)$ is an uncountable group then there is no σ-finite invariant universal measure on $(G, \circ)$. Kannan and Raju [3] asked whether every invariant semiregular universal measure on a group is necessarily a multiple of the counting measure. It is clear that any measure providing a counterexample has to vanish on singletons.

We give the following solution to the above problem.

THEOREM 1. *The following are equivalent*:

($*$) *Every universal invariant semiregular measure on a group is a multiple of the counting measure.*

($**$) *There does not exist a real valued measurable cardinal.*

PROOF.[2] Assume that ($*$) does not hold. Hence there exists a universal invariant semiregular measure on a group, vanishing on singletons. Let A be any set of

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[1]According to the common habit in modern set theory we identify every ordinal with the set of its predecessors and cardinals with initial ordinals.

[2]The present version of the proof is simpler than the original one thanks to the referee's suggestion.

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positive finite measure. It is well known (cf. [**1**]) that the minimal cardinality of a subset $B \subset A$ of positive measure is a real-valued measurable cardinal. Assume that (**) is false and let κ be a real-valued measurable cardinal with a κ-additive universal measure μ on κ vanishing on singletons and such that $\mu(\kappa) = 1$. Let $(G, \circ)$ be the direct sum of κ copies of the two-element group C_2. It will be convenient to view $(G, \circ)$ as the group of all finite subsets of κ with symmetric difference as the group operation.

For each $p \in G$ set $G_p = \{q \in G\colon \exists\beta > \max(p)\,[q = p \cup \{\beta\}]\}$. Notice that the sets G_p together with $\{\varnothing\}$ form a partition of G. Define $m\colon \mathcal{P}(G) \to R \cup \{+\infty\}$ by

$$m(A) = \sum_{p\in G} \mu\left(\bigcup\left[G_p \cap A\right]\right),$$

the right-hand sum being the upper bound of appropriate sums over finite subsets of G.

Clearly $m(\{p\}) = 0$ for each $p \in G$. If $m(A) > 0$ then there is some $p \in G$ such that $\mu(\bigcup[G_p \cap A]) > 0$. Of course, $A \cap G_p \subset A$ and $0 < m(A \cap G_p) = \mu(\bigcup[G_p \cap A]) < \infty$. Hence m is semiregular.

We check that m is κ-additive. Suppose $\{A_\gamma\colon \gamma < \beta\}$ is a pairwise disjoint family of subsets of G, with $\beta < \kappa$. Then for every $p \in G$ the sets $\bigcup(G_p \cap A_\gamma)\colon \gamma < \beta$ are pairwise almost disjoint and we get

$$\begin{aligned} m\left(\bigcup_{\gamma<\beta} A_\gamma\right) &= \sum_{p\in G} \mu\left(\bigcup\left[G_p \cap \left(\bigcup_{\gamma<\beta} A_\gamma\right)\right]\right) \\ &= \sum_{p\in G} \mu\left(\bigcup_{\gamma<\beta} \bigcup\left[G_p \cap A_\gamma\right]\right) \\ &= \sum_{\gamma<\beta} \sum_{p\in G} \mu\left(\bigcup\left[G_p \cap A_\gamma\right]\right) = \sum_{\gamma<\beta} m(A_\gamma). \end{aligned}$$

We now check that m is invariant. Let $q \in G$ and $A \subset G$.

$$\begin{aligned} m(q \circ A) &= \sum_{p\in G} \mu\left(\bigcup\left[G_p \cap q \circ A\right]\right) \\ &= \sum_{p\in G} \mu\left(\bigcup\left[G_{p\circ q} \cap q \circ A\right]\right). \end{aligned}$$

For every $p \in G$ and $\beta > \max(p \circ q)$ we have

$$\begin{aligned} &\beta \in \bigcup\left[G_p \cap A\right] \equiv p \cup \{\beta\} \in A \\ &\equiv q \circ (p \cup \{\beta\}) \in q \circ A \equiv (q \circ p) \cup \{\beta\} \in q \circ A \\ &\equiv \beta \in \bigcup\left[G_{p\circ q} \cap q \circ A\right]. \end{aligned}$$

Hence in view of κ-additivity of μ we get

$$\mu\left(\bigcup\left[G_p \cap A\right]\right) = \mu\left(\bigcup\left[G_{p\circ q} \cap q \circ A\right]\right)$$

and finally

$$m(q \circ A) = \sum_{p\in G} \mu\left(\bigcup\left[G_p \cap A\right]\right) = m(A).$$

This shows that m is an invariant measure and hence a counterexample to (*) which concludes the proof of our theorem.

An extension of the above argument gives the following

THEOREM 2. *Let $(G, +)$ be an abelian group. Then there exists a universal semiregular invariant measure on $(G, +)$ vanishing on singletons iff there exists a real-valued measurable cardinal $\kappa \leqslant |G|$.*

PROOF. If there exists a measure with the above properties on $(G, +)$ then, as before, there is a subset of G with real-valued measurable cardinality.

In order to show the converse we first consider the case when $(G, +)$ is torsion-free and $|G| = \kappa$ is itself real-valued measurable. Let μ be a κ-additive universal finite measure on κ, vanishing on singletons. We fix a well-ordering z_i: $i \in \omega$ of all nonzero integers and for $g \in G$ denote $z_i g$ the result of adding $g\,|z_i|$ times if $z_i > 0$ and adding $-g\,|z_i|$ times if $z_i < 0$.

The group G has a free abelian subgroup G' with base of cardinality κ. Let x_α: $\alpha < \kappa$ be an enumeration of this base. Every element $g \in G'$ has exactly one representation as $z_{i_1} x_{\alpha_1} + \cdots + z_{i_n} x_{\alpha_n}$.

Denote by $\varphi(g)$ the set $\{\alpha_1, \ldots, \alpha_n\} \subset \kappa$. Following the idea from the proof of Theorem 1 we put

$$G_g^i = \left\{h \in G' : \exists\beta > \max(\varphi(g))\left[h = z_i x_\beta + g\right]\right\},$$

for $g \in G'$, $i \in \omega$. Now the measure m: $\mathcal{P}(G') \to R \cup \{+\infty\}$ is defined as

$$m(A) = \sum_{i \in \omega} \sum_{g \in G'} \mu\Big(\bigcup \left(\varphi * \left[G_g^i \cap A\right]\right)\Big).$$

The verification that m has all required properties is similar to that in the proof of Theorem 1.

In order to define the measure m_1 on G we let $\{s_\alpha + G' : \alpha < \lambda\}$ be the family of cosets of G' in G ($\{s_\alpha : \alpha < \lambda\}$ is a fixed selector of this family). On each coset we define m_1 separately putting $m_1(s_\alpha + A) = m(A)$ where $A \subset G'$. Since every $A \subset G$ splits into sets $s_\alpha + A_\alpha$: $\alpha < \lambda$ where $A_\alpha \subset G'$ we can define $m_1(A) = \Sigma_{\alpha<\lambda}\, m(A_\alpha)$.

It is easy to see that m_1 is a κ-additive semiregular universal measure vanishing on singletons. The only property to verify is invariance. By definition it is enough to check $m_1(g + B) = m_1(B)$ for $g \in G$ and B included in a coset. Let $g = s_\alpha + a$, $a \in G'$, and $B = s_\beta + A$, $A \subset G'$. We get $m_1(g + B) = m_1(s_\alpha + s_\beta + a + A)$. There exist an element $b \in G'$ and an ordinal $\gamma < \lambda$ such that $s_\alpha + s_\beta = s_\gamma + b$. Hence $m_1(s_\alpha + s_\beta + a + A) = m_1(s_\gamma + b + a + A) = m(b + a + A)$. In view of the invariance of m we have $m(b + a + A) = m(A)$ and by definition $m_1(B) = m(A)$. Hence finally $m_1(g + B) = m_1(B)$. This finishes the proof in the case when $(G, +)$ is a torsion-free abelian group of cardinality κ,

Next we consider the case of arbitrary abelian groups $(G, +)$ of cardinality κ. Let H be the torsion subgroup of G. First assume that $|H| = \kappa$. Then for some $n \in \omega$ the group H_n of elements of rank dividing n has cardinality κ. Let n_0 be the smallest n with this property. We claim that n_0 is prime. If not, let $n_0 = n_1 n_2$, $1 < n_1, n_2$.

There exist κ elements a_α: $\alpha < \kappa$ of rank exactly n_0. Consider the family $\{n_1 \cdot a_\alpha$: $\alpha < \kappa\}$. All these elements have rank n_2 hence there are $< \kappa$ of them. It follows that there are κ distinct elements which have equal n_1-multiples.

Thus there are κ elements with rank dividing n_1 contrary to the minimality of n_0.

This proves that n_0 is a prime number and hence the set consisting of elements of rank n_0 is (together with the neutral element) a group of cardinality κ. It is also a linear space over the field Z_{n_0}. Hence we can construct a universal semiregular invariant measure m_2 on H_{n_0} similarly as on a free abelian group. Next we define an extension of this measure to the whole group $(G, +)$ first separately on cosets of H_{n_0} in G and then for an arbitrary set as the sum of measures of its intersections with cosets. All details are much the same as in the torsion-free case and hence we leave them to the reader.

If the torsion subgroup H has cardinality $< \kappa$ then G/H is a torsion-free abelian group of cardinality κ. Hence, by the first part of the proof there exists a universal semiregular invariant measure m_3 on G/H vanishing on singletons.

Let $U = \{u_\alpha$: $\alpha < \kappa\}$ be any selector of cosets of H in G. The sets $h + U$: $h \in H$ form a disjoint partition of G. On each of them we define the measure m_4 separately: for $A \subset U$ and $h \in H$, $m_4(h + A) = m_3(\{a + H\colon a \in A\})$.

Finally for an arbitrary $A \subset G$ we put $m_4(A) = \Sigma_{h \in H} m_4(A_h)$ where $A_h = A \cap (h + U)$. Again it is not hard to see that m_4 is universal, κ-additive, semiregular and vanishes on singletons. The only property to verify is invariance. In view of κ-additivity of m_4 and of $|H| < \kappa$ it is enough to check $m_4(a + A) = m_4(A)$, for $a \in G$, and $A \subset U$. The set A splits into disjoint sets $\tilde{A}_k$: $h \in H$, such that $a + \tilde{A}_h \subset h + U$. Again in view of κ-additivity it is enough to check $m_4(a + \tilde{A}_h) = m_4(\tilde{A}_h)$ for each $h \in H$ separately. By definition of m_4 and invariance of the measure m_3 we get

$$\begin{aligned} m_4(a + \tilde{A}_h) &= m_3(\{b + H\colon b \in a + \tilde{A}_h\}) \\ &= m_3(\{-a + b + H\colon b \in a + \tilde{A}_h\}) = m_3(\{b + H\colon b \in \tilde{A}_h\}) = m_4(\tilde{A}_h). \end{aligned}$$

This proves the invariance of m_4 and finishes the proof in the case when $(G, +)$ is an abelian group with real-valued measurable cardinality.

In the general case let $\kappa \leqslant |G|$ be a real-valued measurable cardinal and let H be a subgroup of G of cardinality κ. We construct a measure on H with the required properties and then extend it to G as described in the previous part of the proof.

REMARK. The above constructed measure has the following additional property: any set A has the same measure as $\{-a\colon a \in A\}$.

PROBLEM. Is Theorem 2 true for arbitrary groups?

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW, PKiN IXp, 00-901 WARSAW, POLAND