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ON UNICITY OF COMPLEX POLYNOMIAL L_1-APPROXIMATION ALONG CURVES

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ABSTRACT. We study the unicity of best polynomial L_1-approximation of complex continuous functions along curves γ in the complex plane. A sufficient condition on γ is given which implies unicity. In particular our result includes the known cases of circle and line segment.

Let γ be a curve in the complex plane $\mathbf{C}$ given by

$$z(\varphi) = \rho(\varphi)e^{i\varphi} \qquad (|\varphi| \leqslant \varphi^*), \tag{1}$$

where $\rho(\varphi)$ is a real positive function satisfying the Lipschitz condition, $0 < \varphi^* \leqslant \pi$. Curves of this type will be called admissible. Further, let $C_1(\gamma)$ be the space of complex continuous functions on γ endowed with the L_1-norm:

$$\|f\| = \int_\gamma |f(z)|\, ds; \qquad P_n = \left\{ \sum_{k=0}^{n} c_k z^k,\, c_k \in \mathbf{C} \right\} \quad (n \geqslant 1)$$

denotes the set of polynomials of degree at most n. We say that $p_n \in P_n$ is a best approximation of $f \in C_1(\gamma)$ if $\|f - p_n\| = \inf\{\|f - q_n\|: q_n \in P_n\}$. If each $f \in C_1(\gamma)$ possesses a unique best approximation out of P_n then P_n is called unicity subspace of $C_1(\gamma)$. In the present note we shall study the following problem: which properties of γ guarantee that P_n is a unicity subspace of $C_1(\gamma)$? (Remark that the above property of γ is independent of any linear transformation $z' = az + b$.)

It is known that unicity holds in two classical situations: γ is an arc of circle (Havinson [1]) or a line segment (Kripke-Rivlin [2]). In this note we shall establish a class of curves which guarantee uniqueness. This class will contain among others the circle and the line segment.

For a curve γ given by (1) we set $\rho_0(\gamma) = \min_{|\varphi| \leqslant \varphi^*} \rho(\varphi) > 0$,

$$M(\gamma) = \sup\left\{ \left| \frac{\rho(\varphi_1) - \rho(\varphi_2)}{\varphi_1 - \varphi_2} \right| : |\varphi_1|, |\varphi_2| \leqslant \varphi^*, \varphi_1 \neq \varphi_2 \right\},$$

$$M^*(\gamma) = \sup\left\{ \left| \frac{\rho(\varphi_1) - \rho(\varphi_2)}{2 \sin \frac{\varphi_1 - \varphi_2}{2}} \right| : |\varphi_1|, |\varphi_2| < \varphi^*, \varphi_1 \neq \varphi_2 \right\}.$$

Here $M(\gamma)$ is the usual Lipschitz constant of ρ, $M^*(\gamma)$ is the "periodic" Lipschitz constant of ρ. Evidently, $M(\gamma) \leqslant M^*(\gamma)$.

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Our principal result is the following

THEOREM. *Let γ be an admissible curve satisfying relation*

$$M^*(\gamma) \leqslant \frac{\ln 2}{n+1}\rho_0(\gamma). \tag{2}$$

Then P_n is a unicity subspace of $C_1(\gamma)$.

For an arc of circle with center at origin $M^*(\gamma) = 0$, i.e. the result of Havinson immediately follows. Moreover, transposing a line segment to "infinity" we can achieve that $M^*(\gamma)$ is as small as we like. Thus the result of Kripke-Rivlin is also included in our theorem. At first we shall prove the theorem and then give some corollaries and examples.

We start with a lemma giving a characterization of unicity subspaces of $C_1(\gamma)$. It is a special case of a more general statement verified in [**3**, Theorem 2.9, p. 235]. As usual, for $a \in \mathbf{C}$ sign $a = \bar{a}/|a|$ if $a \neq 0$ and sign $a = 0$ if $a = 0$.

LEMMA. *The following statements are equivalent*:

(i) *P_n is a unicity subspace of $C_1(\gamma)$*;

(ii) *there do not exist $p_n \in P_n \setminus \{0\}$ and $p^* \in C_1(\gamma)$ such that $p^*(z) = \pm p_n(z)$ for each $z \in \gamma$ and*

$$\int_\gamma q_n \operatorname{sign} p^* \, ds = 0 \tag{3}$$

for any $q_n \in P_n$.

PROOF OF THE THEOREM. Assume that (2) holds but P_n is not a unicity subspace of $C_1(\gamma)$. Then by the lemma there exist $p_n \in P_n \setminus \{0\}$ and $p^* \in C_1(\gamma)$ such that $p^*(z) = \pm p_n(z)$ for any $z \in \gamma$ and (3) holds for arbitrary $q_n \in P_n$. Evidently,

$$p^*(z) = \alpha(z) p_n(z), \tag{4}$$

where $\alpha(z) = 1$ or -1 and $\alpha(z)$ is continuous at each point of γ where p_n does not vanish. Then setting $\alpha^*(\varphi) = \alpha(\rho(\varphi)e^{i\varphi})$, we can choose points $\{\varphi_j\}_{j=1}^k$, $-\varphi^* = \varphi_0 < \varphi_1 < \varphi_2 < \cdots < \varphi_k < \varphi_{k+1} = \varphi^*$ such that

$$\alpha^*(\varphi) = \eta(-1)^{j-1} \tag{5}$$

while $\varphi \in (\varphi_{j-1}, \varphi_j)$ $(1 \leqslant j \leqslant k+1,\ \eta = \pm 1)$. Evidently $p_n(z_j) = 0$, where $z_j = \rho(\varphi_j)e^{i\varphi_j}$, $1 \leqslant j \leqslant k$. Hence $k \leqslant n$. (If $k = 0$, then p^* is a polynomial, which contradicts (3).) We shall consider separately the cases of even and odd k.

Case A. $k = 2p$, $p \in \mathbf{N}$. Set $\delta = \frac{1}{2}\sum_{j=1}^k \varphi_j$,

$$q_n^* = (-1)^p \eta \frac{e^{i\delta} z^p p_n}{\prod_{j=1}^k (z - z_j)}.$$

Since p_n vanishes at z_j $(1 \leqslant j \leqslant k)$, $q_n^* \in P_n$. Using (4) we obtain

$$\begin{aligned} q_n^* \operatorname{sign} p^* &= (-1)^p \eta \frac{e^{i\delta} z^p \alpha(z) |p_n|}{\prod_{j=1}^k (z - z_j)} \\ &= (-1)^p \frac{|p_n|}{|\prod_{j=1}^k (z - z_j)|^2} \eta e^{i\delta} z^p \alpha(z) \prod_{j=1}^k (\bar{z} - \bar{z}_j). \end{aligned} \tag{6}$$

Setting $z = \rho(\varphi)e^{i\varphi}$ we have

(7)

$$\begin{aligned}\eta e^{i\delta} z^p \alpha(z) \prod_{j=1}^{k} (\bar{z} - \bar{z}_j) &= \eta \rho^p(\varphi)\alpha^*(\varphi) \prod_{j=1}^{k} \left(\rho(\varphi) e^{i(\varphi_j - \varphi)/2} - \rho(\varphi_j) e^{i(\varphi - \varphi_j)/2}\right) \\ &= \eta \rho^p(\varphi)\alpha^*(\varphi) \prod_{j=1}^{k} \left\{\left(\rho(\varphi) - \rho(\varphi_j)\right)\cos\frac{\varphi_j - \varphi}{2} + i\left(\rho(\varphi) + \rho(\varphi_j)\right)\sin\frac{\varphi_j - \varphi}{2}\right\} \\ &= \eta\alpha^*(\varphi)F(\varphi) \prod_{j=1}^{k} \left(a_j(\varphi) + i\right),\end{aligned}$$

where

$$F(\varphi) = \rho^p(\varphi) \prod_{j=1}^{k} \left(\rho(\varphi) + \rho(\varphi_j)\right) \prod_{j=1}^{k} \sin\frac{\varphi_j - \varphi}{2}, \tag{8}$$

$$a_j(\varphi) = \frac{\left(\rho(\varphi) - \rho(\varphi_j)\right)\cos\left[(\varphi_j - \varphi)/2\right]}{\left(\rho(\varphi) + \rho(\varphi_j)\right)\sin\left[(\varphi_j - \varphi)/2\right]} \qquad (1 \leqslant j \leqslant k). \tag{9}$$

From (5) and (8) one can easily derive that $\eta\alpha^*(\varphi)F(\varphi) = |F(\varphi)|$ for each $|\varphi| \leqslant \varphi^*$. Hence (6) and (7) and yield that

$$\begin{aligned}q_n^* \operatorname{sign} p^* &= (-1)^p \frac{|p_n||F(\varphi)|}{|\prod_{j=1}^{k}(z - z_j)|^2} \prod_{j=1}^{k} \left(a_j(\varphi) + i\right) \\ &= (-1)^p C(\varphi) \prod_{j=1}^{k} \left(a_j(\varphi) + i\right),\end{aligned} \tag{10}$$

where $C(\varphi) > 0$ a.e. on $(-\varphi^*, \varphi^*)$. Moreover by (9) and (2) for each $1 \leqslant j \leqslant k$ and $\varphi \neq \varphi_j$

$$|a_j(\varphi)| < \frac{1}{\rho_0(\gamma)} \left| \frac{\rho(\varphi) - \rho(\varphi_j)}{2\sin\left[(\varphi_j - \varphi)/2\right]} \right| \leqslant \frac{M^*(\gamma)}{\rho_0(\gamma)} \leqslant \frac{\ln 2}{n+1}. \tag{11}$$

Furthermore,

$$\begin{aligned}\prod_{j=1}^{k} \left(a_j(\varphi) + i\right) &= (-1)^p + \sum_{s=1}^{k} \sum_{1 \leqslant j_1' < \cdots < j_s' \leqslant k} a_{j_1'}(\varphi) \cdot \ldots \cdot a_{j_s'}(\varphi) i^{k-s} \\ &= (-1)^p + K(\varphi).\end{aligned} \tag{12}$$

Thus it follows from (11) that for any $\varphi \neq \varphi_j$, $1 \leqslant j \leqslant k$,

$$\begin{aligned}|K(\varphi)| &< \sum_{s=1}^{k} \binom{k}{s} \left(\frac{\ln 2}{n+1}\right)^s = \left(1 + \frac{\ln 2}{n+1}\right)^k - 1 \\ &\leqslant \left(1 + \frac{\ln 2}{n+1}\right)^n - 1 \leqslant e^{\ln 2} - 1 = 1.\end{aligned}$$

Therefore using (10) and (12) we obtain

$$\operatorname{Re} q_n^* \operatorname{sign} p^* = C(\varphi)\operatorname{Re}(-1)^p \prod_{j=1}^{k} \left(a_j(\varphi) + i\right)$$

$$= C(\varphi)\left(1 + \operatorname{Re}(-1)^p K(\varphi)\right) > 0$$

a.e. on $(-\varphi^*, \varphi^*)$. But we have shown that (3) holds for each $q_n \in P_n$, i.e. we arrived at a contradiction.

Case B. $k = 2p + 1$, $p \in \mathbf{Z}_+$. Since this case is rather similar to the Case A we shall only outline the proof.

Set

$$\tilde{q}_n = \eta(-1)^p \frac{e^{i\delta'} z^p (z - z^*) p_n}{\prod_{j=1}^{k}(z - z_j)},$$

where $z^* = \rho(\varphi^*)e^{i\varphi^*}$, $\delta' = \frac{1}{2}\sum_{j=1}^{k}\varphi_j - \varphi^*/2$. Obviously, $\tilde{q}_n \in P_n$. Analogously to the Case A we can show that

$$\tilde{q}_n \operatorname{sign} p^* = (-1)^{p+1} C_1(\varphi) \prod_{j=1}^{k+1} \left(a_j(\varphi) + i\right),$$

where $C_1(\varphi) = C(\varphi)(\rho(\varphi) + \rho(\varphi^*))\,|\sin[(\varphi - \varphi^*)/2]|$, $C(\varphi)$ and $a_j(\varphi)$, $1 \leqslant j \leqslant k$, are as in the Case A and

$$a_{k+1}(\varphi) = \frac{(\rho(\varphi) - \rho(\varphi^*))\cos[(\varphi - \varphi^*)/2]}{(\rho(\varphi) + \rho(\varphi^*))\sin[(\varphi - \varphi^*)/2]}.$$

Then similarly to the Case A we can show that $\operatorname{Re} \tilde{q}_n \operatorname{sign} p^* > 0$ a.e. on $(-\varphi^*, \varphi^*)$, contradicting (3). The proof of the theorem is completed.

Let us give some corollaries in terms of usual Lipschitz constant.

Recall that the curve γ is called closed if $\varphi^* = \pi$ and $\rho(-\pi) = \rho(\pi)$. It can be easily shown that for a closed curve $M^*(\gamma) \leqslant (\pi/2)M(\gamma)$. This immediately leads to the following

COROLLARY 1. *Let γ be an arc of a closed admissible curve γ' satisfying relation*

$$M(\gamma') \leqslant \frac{2}{\pi}\,\frac{\ln 2}{n+1}\rho_0(\gamma').$$

Then P_n is a unicity subspace of $C_1(\gamma)$.

Moreover, if $\varphi^* < \pi$ we have an obvious estimation $M^*(\gamma) \leqslant \varphi^* M(\gamma)/\sin\varphi^*$ implying

COROLLARY 2. *Let γ be an admissible curve such that $\varphi^* < \pi$ and*

$$M(\gamma) \leqslant \frac{\sin\varphi^*}{\varphi^*}\,\frac{\ln 2}{n+1}\rho_0(\gamma). \tag{13}$$

Then P_n is a unicity subspace of $C_1(\gamma)$.

Evidently, Corollary 1 includes the case of circle, while Corollary 2 includes the case of line segment, since a line segment transposed "far enough" satisfies (13).

More generally we can get the following

COROLLARY 3. *Let the curve* γ *be given by*

$$z(y) = f(y) + iy \qquad (y \in [-h, h], h > 0),$$

where f is a real Lipschitzian function satisfying relation

$$M(f) < \frac{\ln 2}{n+1}. \tag{14}$$

Then P_n *is a unicity subspace of* $C_1(\gamma)$.

PROOF. For a given $c > 0$ we consider the curve γ_c given by

$$z(y) = c + f(y) + iy \qquad (|y| \leqslant h),$$

i.e. $z(\varphi) = \rho_c(\varphi)e^{i\varphi}$, $\varphi \in (\varphi'(c), \varphi''(c))$. Evidently,

$$\varphi^* = \varphi^*(c) = \max\{|\varphi'(c)|, |\varphi''(c)|\} \to 0 \qquad (c \to \infty).$$

Let us prove that $\rho_c(\varphi)$ is a single-valued function for c large enough. If $\rho_j e^{i\varphi} \in \gamma_c$, $j = 1, 2$, then

$$(\rho_1 - \rho_2)\cos\varphi = f(\rho_1 \sin\varphi) - f(\rho_2 \sin\varphi).$$

Therefore

$$|\rho_1 - \rho_2|\cos\varphi^* \leqslant M(f)|\rho_1 - \rho_2||\sin\varphi^*|.$$

Since $\varphi^* \to 0$ $(c \to \infty)$, $\rho_1 = \rho_2$. Thus for c large enough γ_c is admissible. Let us estimate $M(\gamma_c)$. Set $\rho_j = \rho_c(\varphi_j)$, $\varphi_j \in (\varphi', \varphi'')$ $(j = 1, 2)$. Then

$$\rho_1 \cos\varphi_1 - \rho_2 \cos\varphi_2 = f(\rho_1 \sin\varphi_1) - f(\rho_2 \sin\varphi_2). \tag{15}$$

Estimating the left and right parts of this equality we obtain

$$|\rho_1 \cos\varphi_1 - \rho_2\cos\varphi_2| \geqslant |\rho_1 - \rho_2|\cos\varphi^* - \rho_1|\varphi_1 - \varphi_2||\sin\varphi^*|,$$

$$|f(\rho_1\sin\varphi_1) - f(\rho_2\sin\varphi_2)| \leqslant M(f)\rho_1|\varphi_1 - \varphi_2| + M(f)|\rho_1 - \rho_2||\sin\varphi^*|.$$

Hence and by (15)

$$\begin{aligned} |\rho_1 - \rho_2|\left(\cos\varphi^* - M(f)|\sin\varphi^*|\right) &\leqslant \rho_1|\varphi_1 - \varphi_2|\left(M(f) + |\sin\varphi^*|\right) \\ &\leqslant \left(\rho_0(\gamma_c) + 2M(\gamma_c)\varphi^*\right)\left(M(f) + |\sin\varphi^*|\right)|\varphi_1 - \varphi_2|. \end{aligned}$$

By simple calculations we arrive at

$$\limsup_{c\to\infty} \frac{M(\gamma_c)}{\rho_0(\gamma_c)} \leqslant M(f).$$

Finally, using (14) we obtain that for c large enough γ_c satisfies the conditions of Corollary 2. Thus P_n is a unicity subspace of $C_1(\gamma_c)$, i.e. a unicity subspace of $C_1(\gamma)$, too.

Now we can obtain the case of line segment simply setting in Corollary 3 $f(y) \equiv \text{const}$.

Let us give some examples of curves γ different from the circle and the line segment, for which P_n is a unicity subspace of $C_1(\gamma)$. The proofs follow by easy calculations from Corollaries 1–3.

EXAMPLE 1. γ is an ellipse given by $z(\varphi) = \sqrt{a^2\cos^2\varphi + \sin^2\varphi}\, e^{i\varphi}$ $(|\varphi| \leqslant \pi)$, where $1 \leqslant a \leqslant \sqrt{1 + 2\ln 2/(n+1)}$.

EXAMPLE 2. γ is a polygon inscribed into the unit circle with sides of length less than $\pi/2(n+1)$.

EXAMPLE 3. γ is given by $z(x) = x + if(x)$ $(|x| \leqslant 1)$, where f is a piecewise linear real continuous function such that the slope of each of its linear parts is less than $\ln 2/(n+1)$ (in absolute value).

Finally, we would like to mention some open problems. The first question is connected with sharpness of estimation (2). An example of curve for which uniqueness fails was given in [**2**], but that curve even is not admissible in our sense. The second problem can be posed as follows: Are there any other curves beside circle and line segment which imply unicity of polynomial L_1-approximation for *each n*? Actually, our theorem extends the class of curves which guarantee uniqueness only for arbitrary *given n*.

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