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ON SETS OF MUTUALLY DISJOINT UNIVALENT MEROMORPHIC FUNCTIONS

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ABSTRACT. Several conditions are established for a set of meromorphic functions in a given domain D, in the complete complex plane, to consist of mutually disjoint univalent functions in terms of a certain class of differential operators which are invariant under the Möbius group.

1. Introduction. Let $\mathfrak{M}^n(D)$ denote the set of all n-tuples $(f_1, f_2, \ldots, f_n)$ of mutually disjoint meromorphic functions in a domain D in $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, and let $\mathfrak{M}_0^n(D)$ be the subset of all $(f_1, f_2, \ldots, f_n) \in \mathfrak{M}^n(D)$ such that each f_i, $1 \leq i \leq n$, is univalent in D. Let D be a domain which is not in O_{AD}, i.e. D supports nonconstant functions with $L^2(D)$-derivatives. In previous papers **[7, 8, 9]** we have studied necessary and sufficient conditions for an element $f \in \mathfrak{M}(D) = \mathfrak{M}^1(D)$ to be in $\mathfrak{M}_0(D) = \mathfrak{M}_0^1(D) = \{$univalent functions in $D\}$ in terms of the functions $S_f(z, \zeta)$ and $\psi_m(f, z)$, where

$$(1.1)\quad S_f(z,\zeta) = S_{f,f}(z,\zeta) = \frac{f'(z)f'(\zeta)}{(f(z)-f(\zeta))^2} - \frac{1}{(z-\zeta)^2}, \qquad (z,\zeta) \in D \times D,$$
$$= \sum_{m=2}^{\infty} (m-1)\psi_m(f,z)(\zeta - z)^{m-2}, \qquad z \in D, |\zeta - z| < d(z, \partial D).$$

Here we study similar conditions for an element $(f_1, f_2, \ldots, f_n) \in \mathfrak{M}^n(D)$ to be in $\mathfrak{M}_0^n(D)$, in terms of the functions $S_{f_i f_j}(z,\zeta)$ and $\psi_m(f_i, f_j; z)$ where, for $i \neq j$, we denote

$$(1.2)\quad S_{f_i f_j}(z,\zeta) = \frac{f_i'(z)f_j'(\zeta)}{(f_i(z)-f_j(\zeta))^2}, \qquad (z,\zeta) \in D \times D,$$
$$= \sum_{m=2}^{\infty} (m-1)\psi_m(f_i, f_j; z)(\zeta - z)^{m-2}, \qquad z \in D, |\zeta - z| < d(z, \partial D),$$

and for $i = j$ $S_{f_i f_i}(z,\zeta) = S_{f_i}(z,\zeta)$, $\psi_m(f_i, f_i; z) = \psi_m(f_i, z)$ are defined in (1.1).

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2. Area theorem and $S_{f_if_j}$. Let $(f_1, f_2, \ldots, f_n) \in \mathfrak{M}^n(D)$ and denote $S_{ij} = S_{f_if_j}$, $S_{ii} = S_i = S_{f_i}$. With every domain D which is not in O_{AD} we associate the functions

$$\Gamma_D(z) = \Gamma(z) = \frac{1}{\pi^2} \iint_{\hat{\mathbf{C}} \setminus D} \frac{d\xi\, d\eta}{|\zeta - z|^4}, \qquad z \in D, \zeta = \xi + i\eta, \tag{2.1}$$

the Bergman kernel-function $K_D(z, \bar{\zeta}) = K(z, \bar{\zeta})$ and the Poincaré metric $\rho_D(z) = \rho(z)$ in D (with constant curvature -4). By Burbea and Beardon and Gehring [**6**, **4**] and Bergman and Schiffer's [**5**] results we have

$$\Gamma(z) \leqslant K(z, \bar{z}) \leqslant \frac{1}{\pi} \rho(z)^2, \qquad z \in D. \tag{2.2}$$

We also abbreviate $D_i = f_i(D)$, $\Gamma_i = \Gamma_{D_i}$, $\rho_i = \rho_{D_i}$, etc.

In [**9**] it has been proved:

THEOREM A. *Let $f \in \mathfrak{M}_0(D) = \mathfrak{M}_0^1(D)$. Then*

$$\rho(z)^{-1} \left(\frac{1}{\pi} \iint_D |S_f(z, \zeta)|^2 \, d\xi\, d\eta \right)^{1/2} \leqslant \beta(z; D) + \beta(f(z); f(D)) \leqslant 2 \tag{A.1}$$

where $\beta(z; D) = (1 - \Gamma(z)/K(z, \bar{z}))^{1/2} \leqslant 1$. Conversely, if

$$\rho(z)^{-1} \rho(w)^{-1} |S_f(z, w)| \leqslant K < \infty, \qquad (z, w) \in D \times D, \tag{A.1$'$}$$

for $f \in \mathfrak{M}(D) = \mathfrak{M}^1(D)$, then $f \in \mathfrak{M}_0(D)$.

The following is the analogous result in $\mathfrak{M}^n(D)$.

THEOREM 1. *Let $(f_1, f_2, \ldots, f_n) \in \mathfrak{M}^n(D)$.*

(a) *If $(f_1, f_2, \ldots, f_i, \ldots, f_n) \in \mathfrak{M}_0^{n-1}(D)$ and f_i is locally univalent in D, then*

$$\frac{1}{\pi} \rho(z)^{-2} \iint_D \sum_{j \neq i} |S_{ij}(z, \zeta)|^2 \, d\xi\, d\eta \leqslant \pi \frac{\Gamma_i(f_i(z))}{\rho_i(f_i(z))^2} \leqslant 1, \qquad z \in D. \tag{2.3}$$

(b) *If $(f_1, f_2, \ldots, f_n) \in \mathfrak{M}_0^n(D)$, then*

(2.3′)

$$\rho(z)^{-1} \left(\frac{1}{\pi} \iint_D \sum_{j=1}^{n} |S_{ij}(z, \zeta)|^2 \, d\xi\, d\eta \right)^{1/2} \leqslant 1 + \beta(z, D), \qquad z \in D, 1 \leqslant i \leqslant n.$$

(c) *Let $(f_1, f_2, \ldots, f_n)$ be an n-tuple of meromorphic functions in D. If there is a constant K, such that for all $(z, w) \in D \times D$ and every $1 \leqslant i \leqslant n$, we have*

$$\rho(z)^{-1} \rho(w)^{-1} \sum_{j=1}^{n} |S_{ij}(z, \zeta)| \leqslant K < \infty, \tag{2.3*}$$

then $(f_1, f_2, \ldots, f_n) \in \mathfrak{M}_0^n(D)$.

PROOF. For a fixed $z \in D$ let $h_i(w) = (f_i(z) - w)^{-1}$ and, for $j \neq i$, $g_{ij}(\zeta) = h_i \circ f_j(\zeta) = (f_i(z) - f_j(\zeta))^{-1}$. Since $f_j(\zeta) \neq f_i(z)$ for every $\zeta \in D$, $g_{ij}(\zeta)$ is analytic in

D, and if $f_j(\zeta)$ is univalent in D, so is $g_{ij}(\zeta)$, and $S_{ij}(z,\zeta) = f_i'(z)g_{ij}'(\zeta)$. Hence, the univalence of $g_{ij}(\zeta)$ yields

$$(2.4)\quad \iint_D \sum_{j\neq i} |S_{ij}(z,\zeta)|^2\, d\xi\, d\eta = |f_i'(z)|^2 \sum_{j\neq i} \iint_D |g_{ij}'(\zeta)|^2\, d\xi\, d\eta$$
$$= |f_i'(z)|^2 \sum_{j\neq i} \text{area}\, g_{ij}(D) = |f_i'(z)|^2 \text{area}\left[\bigcup_{j\neq i} h_i(D_j)\right]$$
$$\leqslant |f_i'(z)|^2 \text{area}\left[h_i(\hat{\mathbf{C}} \setminus D_i)\right] = |f_i'(z)|^2 \iint_{\hat{\mathbf{C}}\setminus D_i} |h_i'(\zeta)|^2\, d\xi\, d\eta$$
$$= |f_i'(z)|^2 \iint_{\hat{\mathbf{C}}\setminus D_i} \frac{d\xi\, d\eta}{|\zeta - f_i(z)|^4} = \pi^2 |f_i'(z)|^2 \Gamma_i(f_i(z)), \qquad z \in D.$$

But since $f_i\colon D \to D_i$ is locally univalent, we have

$$(2.5)\qquad \rho(z) = \rho_i(f_i(z))\,|f_i'(z)|\,, \qquad z \in D.$$

Thus (2.4), (2.5) and (2.2) imply (2.3). Obviously there is an equality in (2.3) iff $\bigcup_{j=1}^n \overline{D}_j = \hat{\mathbf{C}}$, and therefore (2.3) is sharp. (2.3′) follows at once from (2.3) and (A.1).

Conversely, by (A.1′), if (2.3*) holds for some i, $1 \leqslant i \leqslant n$, then $f_i \in \mathfrak{M}_0(D)$; on the other hand, if $f_i(D) \cap f_j(D) \neq \varnothing$, one can find a pair $(z,\zeta) \in D \times D$ such that $f_i'(z)f_j'(\zeta) \neq 0$ but $f_i(z) = f_j(\zeta)$. This contradicts (2.3*) and completes the proof of the theorem.

From the reproduction property of $K_D(z,\bar{\zeta})$ and Cauchy-Schwarz' inequality it follows that every nonconstant $L^2(D)$-integrable analytic function ϕ in D satisfies

$$(2.6)\qquad |\phi(w)|^2 \leqslant K(w,\bar{w})\|\phi\|_{L^2}^2 \leqslant \frac{1}{\pi}\rho(w)^2 \iint_D |\phi(\zeta)|^2\, d\xi\, d\eta.$$

Hence, Theorem 1 implies

COROLLARY 1. *If $(f_1, f_2, \ldots, f_n) \in \mathfrak{M}^n(D)$, $(f_1, \ldots, f_i, \ldots, f_n) \in \mathfrak{M}_0^{n-1}(D)$ and f_i locally univalent in D, then*

$$(2.7)\quad \rho_D(z)^{-2}\rho_D(w)^{-2} \sum_{j\neq i} |S_{ij}(z,w)|^2 \leqslant \pi \frac{\Gamma_i(f_i(z))}{\rho_i(f_i(z))^2} \leqslant 1, \qquad (z,w) \in D \times D.$$

COROLLARY 2. *If $(f_1, f_2, \ldots, f_n) \in \mathfrak{M}_0^n(D)$, then*

$$(2.8)\quad \frac{\rho_D(z)^{-2}}{\pi} \iint_D |S_i(z,\zeta)|^2\, d\xi\, d\eta + \rho_D(z)^{-2}\rho_D(w)^{-2} \sum_{j\neq i} |S_{ij}(z,w)|^2$$
$$\leqslant (1 + \beta(z;D))^2 \leqslant 2^2, \qquad (z,w) \in D \times D,\ 1 \leqslant i \leqslant n,$$

and therefore

$$(2.8')\quad \rho_D(z)^{-2}\rho_D(w)^{-2} \sum_{j=1}^{n} |S_{ij}(z,w)|^2 \leqslant (1 + \beta(z,D))^2 \leqslant 2^2,$$
$$(z,w) \in D \times D,\ 1 \leqslant i \leqslant n.$$

REMARK. For $n=2$ and $D=$ the unit disc U, (2.8) is Aharonov's inequality (cf. [2]);

(2.8*)
$$\frac{\left(1-|z|^2\right)^2}{\pi}\iint\limits_{|\zeta|<1}|S_f(z,\zeta)|^2\,d\xi\,d\eta+\left(1-|z|^2\right)^2\left(1-|w|^2\right)^2|S_{f,g}(z,w)|^2\leqslant 1,$$
$$|z|<1,\ |w|<1,$$

for all pairs $(f,g)\in\mathfrak{M}_0^2(U)$.

Inequality (2.6) with a second application of Cauchy-Schwarz' inequality yields

(2.6′)
$$\rho_D(z)^{-1}\rho_D(w)^{-1}\sum_{j=1}^{n}|S_{ij}(z,w)|$$
$$\leqslant\sqrt{n}\,\rho_D(z)^{-1}\left(\frac{1}{\pi}\iint\limits_{D}\sum_{j=1}^{n}|S_{ij}(z,\zeta)|^2\,d\xi\,d\eta\right)^{1/2}.$$

Hence we have

COROLLARY 3. *Let $f_1,f_2,\ldots,f_n$ be meormorphic functions in D. Then the following are equivalent*:

(a) $(f_1,f_2,\ldots,f_n)\in\mathfrak{M}_0^n(D)$;

(b) $\sup_{z\in D}\rho_D(z)^{-1}\left(\frac{1}{\pi}\iint_D\sum_{j=1}^{n}|S_{ij}(z,\zeta)|^2\,d\xi\,d\eta\right)^{1/2}<\infty$, $1\leqslant i\leqslant n$;

(c) $\sup_{z\in D}\sup_{w\in D}\rho_D(z)^{-1}\rho_D(w)^{-1}\sum_{j=1}^{n}|S_{ij}(z,w)|<\infty$, $1\leqslant i\leqslant n$.

3. The invariants $\psi_n(f_i,f_j;z)$. Applying the power series expansions of $S_{ij}(z,\zeta)$, given in (1.1) and (1.2), and the orthogonality relations: $\iint_{|\zeta|<1}\zeta^m\bar{\zeta}^n\,d\xi\,d\eta=\pi\delta_{n,m}/(n+1)$ in the unit disc U, we conclude from (2.3′)

LEMMA 1. *If $(f_1,f_2,\ldots,f_n)\in\mathfrak{M}_0^n(U)$, then for every $1\leqslant i\leqslant n$,*

$$\text{(3.1)}\qquad\sum_{m=2}^{\infty}(m-1)\sum_{j=1}^{n}|\psi_m(f_i,f_j;0)|^2\leqslant 1\qquad\left(\psi_m(f_i,f_i;0)=\psi_m(f_i,0)\right).$$

The sequences $\{\psi_m(f_i,z)\}_{m=1}^{\infty}$ and $\{\psi_m(f_i,f_j;z)\}_{m=1}^{\infty}$ have the following generating functions (cf. Aharonov [1] and Aharonov and Lavie [3]):

(3.2)
$$\frac{f_i'(z)}{f_i(z)-f_i(\zeta)}-\frac{1}{z-\zeta}=\sum_{m=1}^{\infty}\psi_j(f_i,z)(\zeta-z)^{m-1},$$
$$\frac{f_i'(z)}{f_i(z)-f_j(\zeta)}=\sum_{m=1}^{\infty}\psi_m(f_i,f_j;z)(\zeta-z)^{m-1},\qquad z\in D,\ |\zeta-z|<d(z,\partial D),$$

from which it follows that for every $m\geqslant 2$, $\psi_m(f_if_j;z)$ and $\psi_m(f_i,f_i;z)=\psi_m(f_i,z)$ are invariant in the sense:

$$\text{(3.3)}\quad\psi_m(g\circ f_i,g\circ f_j;z)=\psi_m(f_i,f_j;z)\quad\text{for every Möbius transformation } g.$$

In [**8**] it has been proved:

LEMMA B. (a) *Let* $g \in \mathfrak{M}(D)$ *and* $f \in \mathfrak{M}(\tilde{D})$, $\tilde{D} = g(D)$. *Then*

$$\text{(B.1)}\quad \psi_m(f \circ g; z) = \sum_{k=1}^{m} B_{m,k}(g, z)\psi_k(f, g(z)) + \psi_m(g, z), \qquad z \in D, m \geqslant 1,$$

where $B_{m,k}(g, z) = g'(z)A_{m-1,k-1}(g, z)$ *and*

$$(g(z) - g(\zeta))^k = \sum_{m=k}^{\infty} A_{m,k}(g, z)(\zeta - z)^m.$$

(b) *If* $g(z) = (az + b)/(cz + d)$, $ad - bc = 1$, *then for* $m \geqslant k \geqslant 2$

$$\text{(B.1')}\qquad B_{m,k}(g, z) = \binom{m-2}{k-2}(-c)^{m-k} g'(z)^{(m+k)/2}, \qquad \psi_m(g, z) = 0.$$

Similarly, one can easily prove:

LEMMA 2. *If* $g \in \mathfrak{M}(D)$ *and* $(f_1, f_2) \in \mathfrak{M}^2(\tilde{D})$, $\tilde{D} = g(D)$, *then*

$$\text{(3.4)}\quad \psi_m(f_1 \circ g, f_2 \circ g; z) = \sum_{k=2}^{m} B_{m,k}(g, z)\psi_k(f_1, f_2; g(z)), \qquad z \in D, m \geqslant 1,$$

and if $g(z) = (az + b)/(cz + d)$, $ad - bc = 1$, *then*

$$\text{(3.4')}\quad \psi_m(f_1 \circ g, f_2 \circ g; z) = \sum_{k=2}^{m} \binom{m-2}{k-2}(-c)^{m-k} g'(z)^{(m+k)/2}\psi_k(f_1, f_2; g(z)).$$

Lemmas 1 and 2 imply (cf. [**7** and **8**])

THEOREM 2. (a) *If* $(f_1, f_2, \ldots, f_n) \in \mathfrak{M}_0^n(D)$, *then*

$$\text{(3.5)}\quad \sum_{m=2}^{\infty} (m-1)\, d(z, \partial D)^{2m} \sum_{j=1}^{n} |\psi_m(f_i, f_j; z)|^2 \leqslant 1, \qquad z \in D, 1 \leqslant i \leqslant n.$$

(b) *If* $(f_1, f_2, \ldots, f_n) \in \mathfrak{M}_0^n(U)$, *then*

$$\text{(3.6)}\quad \sum_{m=2}^{\infty} (m-1) \sum_{j=1}^{n} \left| \sum_{l=2}^{m} \binom{m-2}{l-2} (-\bar{\zeta})^{m-l} (1 - |\zeta|^2)^l \psi_l(f_i, f_j; \zeta) \right|^2 \leqslant 1,$$

$$|\zeta| < 1,\ 1 \leqslant i \leqslant n,$$

$$\text{(3.7)}\quad (1 - |z|^2)^m |\psi_m(f_i, f_j; z)| \leqslant p_{m-2}(|z|) = \sum_{k=2}^{m} \binom{m-2}{k-2} \frac{|z|^{m-k}}{\sqrt{k-1}},$$

$$|z| < 1,\ 1 \leqslant i, j \leqslant n,$$

and

$$\text{(3.7')}\quad (1 - |z|^2)^m \sum_{j=1}^{n} |\psi_m(f_i, f_j; z)| \leqslant \sqrt{n}\, p_{m-2}(|z|), \qquad |z| < 1, 1 \leqslant i \leqslant n.$$

PROOF. (a) For any $z \in D$ let $g(\zeta) = r\zeta + z$, where $r = d(z, \partial D)$. Then by (3.4') $\psi_m(f_i \circ g, f_j \circ g; \zeta) = r^m \psi_m(f_i, f_j; r\zeta + z)$ for every $\zeta \in U$, and $(f_1 \circ g, f_2 \circ g, \ldots, f_n \circ g)|_U \in \mathfrak{M}_0^n(U)$. Thus (3.1) implies (3.5).

(b) Similarly (3.1) implies (3.6), using the Möbius self-mapping

$$g(z) = (z + \zeta)/(1 + \bar{\zeta} z)$$

of U and (3.4′). For the proof of (3.7) let $g(\zeta) = (\zeta - z)/(1 - \bar{z}\zeta)$. Then by (B.1′) and (3.4′),

$$(3.8)\quad (1 - |z|^2)^m \psi_m(f_i, f_j; z) = \sum_{k=2}^{m} \binom{m-2}{k-2} \psi_k(f_i \circ g^{-1}, f_j \circ g^{-1}; 0)\bar{z}^{m-k}$$

$$(\text{since } g(z) = 0).$$

Notice also that (3.1) yields

$$(3.9)\qquad \sum_{j=1}^{n} |\psi_m(f_i \circ g^{-1}, f_j \circ g^{-1}; 0)|^2 \leqslant \frac{1}{m-1}, \qquad m \geqslant 2,\ 1 \leqslant i, j \leqslant n,$$

and in particular $|\psi_m(f_i \circ g^{-1}, f_j \circ g^{-1}; 0)| \leqslant 1/\sqrt{m-1}$. This implies (3.7) at once, and also the inequality

$$\begin{aligned}(1 - |z|^2)^m &\left| \sum_{j=1}^{n} \psi_m(f_i, f_j; z) e^{i\theta_j} \right| \\ &\leqslant \sum_{k=2}^{m} \binom{m-2}{k-2} |z|^{m-k} \sum_{j=1}^{n} |\psi_k(f_i \circ g^{-1}, f_j \circ g^{-1}; 0)| \\ &\leqslant \sqrt{n}\, p_{m-2}(|z|),\end{aligned}$$

for every choice of $\theta_1, \theta_2, \ldots, \theta_n$. Now take θ_j such that

$$\psi_m(f_i, f_j; z) e^{i\theta_j} = |\psi_m(f_i, f_j; z)|$$

and the proof of (3.7′) is complete. Q.E.D.

4. The invariants $\phi_n(f_i, f_j; z)$. In this section we consider a second sequence of invariants

$$(4.1)\quad \phi_n(f_1, f_2, z) = \frac{3!}{(n+1)!} \phi_2^{(n-2)}(f_1, f_2; z), \qquad n \geqslant 2,\ (f_1, f_2) \in \mathfrak{M}_0^2(D),$$

where

$$\phi_2(f_1, f_2; z) = \psi_2(f_1, f_2; z) = \frac{f_1'(z) f_2'(z)}{(f_1(z) - f_2(z))^2} = S_{f_1, f_2}(z, z),$$

and formally define

$$(4.2)\quad T_{f_1, f_2}(z, \zeta) = \sum_{n=2}^{\infty} (n-1)\phi_n(f_1, f_2; z)(\zeta - z)^{n-2},$$

$$z \in D,\ |\zeta - z| < d(z, \partial D).$$

(Cf. the definitions of $\phi_n(f, z)$ and $T_f(z, \zeta)$ for a single function $f \in \mathfrak{M}(D)$ in [10].)

THEOREM 3. (a) *Let D be a Jordan domain the boundry $C = \partial D$ of which is a quasicircle. Then there is constant $K = K(C)$, depending only on C, such that*

$$\rho_D = (z)^{-1}\left(\frac{1}{\pi}\iint_D |T_{f_1,f_2}(z,\zeta)|^2\,d\xi\,d\eta\right)^{1/2} \leqslant K(C) \tag{4.3}$$

for every pair $(f_1, f_2) \in \mathfrak{M}_0^2(D)$.

(b) *If $C = \partial U$ is the unit circle, then $K(C) \leqslant 3$.*

PROOF. By Corollary 1, if $(f_1, f_2) \in \mathfrak{M}^2(D)$, f_1 locally univalent and $f_2 \in \mathfrak{M}_0(D)$, then

$$\sup_{z\in D}\rho_D(z)^{-2}|\phi_2(f_1, f_2; z)| \leqslant \sup_{z,w\in D}\rho_D(z)^{-1}\rho_D(w)^{-1}|S_{f_1,f_2}(z,w)| \leqslant 1.$$

Hence, by Theorem 1 in **[10]**, if $C = \partial D$ is a quasicircle, then

$$\rho_D(z)^{-1}\left(\frac{1}{\pi}\iint_D |T_{f_1,f_2}(z,\zeta)|^2\,d\xi\,d\eta\right)^{1/2}$$

$$\leqslant K(C)\sup_{z\in D}\rho_D(z)^{-2}|\phi_2(f_1, f_2; z)| \leqslant K(C)$$

for some constant $K(C)$ depending only on C, and $K(C) \leqslant 3$ if $C = \partial U$ is the unit circle. Q.E.D.

From (4.2), (4.3) and Parseval's identity we conclude

LEMMA 3. *If $(f_1, f_2) \in \mathfrak{M}_0^2(U)$, then*

$$\sum_{n=2}^{\infty}(n-1)|\phi_n(f_1, f_2; 0)|^2 \leqslant K(\partial U)^2 \leqslant 9. \tag{4.4}$$

Computations similar to those appearing in **[10]** yield

LEMMA 4. *For every pair $(f_1, f_2) \in \mathfrak{M}^2(D)$ and $g(z) = (az+b)/(cz+d)$, $ad - bc = 1$,*

$$T_{f_1\circ g, f_2\circ g}(z,\zeta) = T_{f_1,f_2}(g(z), g(\zeta))g'(z)g'(\zeta) \tag{4.5}$$

and

$$\phi_n(f_1\circ g, f_2\circ g; z) = \sum_{l=2}^{n}\binom{n-2}{l-2}(-c)^{n-l}g'(z)^{(n+l)/2}\phi_l(f_1, f_2; g(z)). \tag{4.6}$$

Lemmas 3 and 4 imply the following analogue for Theorem 2.

THEOREM 4. (a) *If $(f_1, f_2) \in \mathfrak{M}_0^2(D)$, then*

$$\sum_{n=2}^{\infty}(n-1)\,d(z,\partial D)^{2n}|\phi_n(f_1, f_2; z)|^2 \leqslant 9, \qquad z \in D. \tag{4.7}$$

(b) *If $(f_1, f_2) \in \mathfrak{M}_0^2(U)$, then*

$$\sum_{n=2}^{\infty}(n-1)\left|\sum_{l=2}^{n}\binom{n-2}{l-2}(-\bar\zeta)^{n-l}(1-|\zeta|^2)^l\phi_l(f_1, f_2; \zeta)\right|^2 \leqslant 0, \qquad |\zeta| < 1, \tag{4.8}$$

and

$$(1-|z|^2)^n|\phi_n(f_1, f_2; z)| \leq 3p_{n-2}(|z|) \tag{4.9}$$

where $p_{n-2}(|z|)$ *is the real polynomial defined in* (3.7).

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