

ON SETS OF MUTUALLY DISJOINT UNIVALENT MEROMORPHIC FUNCTIONS

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ABSTRACT. Several conditions are established for a set of meromorphic functions in a given domain D , in the complete complex plane, to consist of mutually disjoint univalent functions in terms of a certain class of differential operators which are invariant under the Möbius group.

1. Introduction. Let $\mathfrak{N}^n(D)$ denote the set of all n -tuples (f_1, f_2, \dots, f_n) of mutually disjoint meromorphic functions in a domain D in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and let $\mathfrak{N}_0^n(D)$ be the subset of all $(f_1, f_2, \dots, f_n) \in \mathfrak{N}^n(D)$ such that each f_i , $1 \leq i \leq n$, is univalent in D . Let D be a domain which is not in O_{AD} , i.e. D supports nonconstant functions with $L^2(D)$ -derivatives. In previous papers [7, 8, 9] we have studied necessary and sufficient conditions for an element $f \in \mathfrak{N}(D) = \mathfrak{N}^1(D)$ to be in $\mathfrak{N}_0(D) = \mathfrak{N}_0^1(D) = \{\text{univalent functions in } D\}$ in terms of the functions $S_f(z, \zeta)$ and $\psi_m(f, z)$, where

$$(1.1) \quad S_f(z, \zeta) = S_{f,f}(z, \zeta) = \frac{f'(z)f'(\zeta)}{(f(z) - f(\zeta))^2} - \frac{1}{(z - \zeta)^2}, \quad (z, \zeta) \in D \times D,$$

$$= \sum_{m=2}^{\infty} (m-1)\psi_m(f, z)(\zeta - z)^{m-2}, \quad z \in D, |\zeta - z| < d(z, \partial D).$$

Here we study similar conditions for an element $(f_1, f_2, \dots, f_n) \in \mathfrak{N}^n(D)$ to be in $\mathfrak{N}_0^n(D)$, in terms of the functions $S_{f_i f_j}(z, \zeta)$ and $\psi_m(f_i, f_j; z)$ where, for $i \neq j$, we denote

$$(1.2) \quad S_{f_i f_j}(z, \zeta) = \frac{f_i'(z)f_j'(\zeta)}{(f_i(z) - f_j(\zeta))^2}, \quad (z, \zeta) \in D \times D,$$

$$= \sum_{m=2}^{\infty} (m-1)\psi_m(f_i, f_j; z)(\zeta - z)^{m-2}, \quad z \in D, |\zeta - z| < d(z, \partial D),$$

and for $i = j$ $S_{f_i f_i}(z, \zeta) = S_{f_i}(z, \zeta)$, $\psi_m(f_i, f_i; z) = \psi_m(f_i, z)$ are defined in (1.1).

Received by the editors March 12, 1982.

1980 *Mathematics Subject Classification*. Primary 30C35.

Key words and phrases. Mutually disjoint functions, univalent functions, invariants.

¹I wish to thank Professor Uri Srebro for his valuable support and useful remarks. I also wish to thank Professor Dov Aharonov for suggesting the close connection between his work and mine.

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2. Area theorem and S_{f_j} . Let $(f_1, f_2, \dots, f_n) \in \mathfrak{N}^n(D)$ and denote $S_{ij} = S_{f_j f_j}$, $S_{ii} = S_i = S_{f_i}$. With every domain D which is not in O_{AD} we associate the functions

$$(2.1) \quad \Gamma_D(z) = \Gamma(z) = \frac{1}{\pi^2} \iint_{\dot{C} \setminus D} \frac{d\xi d\eta}{|\xi - z|^4}, \quad z \in D, \xi = \xi + i\eta,$$

the Bergman kernel-function $K_D(z, \bar{\zeta}) = K(z, \bar{\zeta})$ and the Poincaré metric $\rho_D(z) = \rho(z)$ in D (with constant curvature -4). By Burbea and Beardon and Gehring [6, 4] and Bergman and Schiffer's [5] results we have

$$(2.2) \quad \Gamma(z) \leq K(z, \bar{z}) \leq \frac{1}{\pi} \rho(z)^2, \quad z \in D.$$

We also abbreviate $D_i = f_i(D)$, $\Gamma_i = \Gamma_{D_i}$, $\rho_i = \rho_{D_i}$, etc.

In [9] it has been proved:

THEOREM A. *Let $f \in \mathfrak{N}_0(D) = \mathfrak{N}_0^1(D)$. Then*

$$(A.1) \quad \rho(z)^{-1} \left(\frac{1}{\pi} \iint_D |S_f(z, \zeta)|^2 d\xi d\eta \right)^{1/2} \leq \beta(z; D) + \beta(f(z); f(D)) \leq 2$$

where $\beta(z; D) = (1 - \Gamma(z)/K(z, \bar{z}))^{1/2} \leq 1$. Conversely, if

$$(A.1') \quad \rho(z)^{-1} \rho(w)^{-1} |S_f(z, w)| \leq K < \infty, \quad (z, w) \in D \times D,$$

for $f \in \mathfrak{N}(D) = \mathfrak{N}^1(D)$, then $f \in \mathfrak{N}_0(D)$.

The following is the analogous result in $\mathfrak{N}^n(D)$.

THEOREM 1. *Let $(f_1, f_2, \dots, f_n) \in \mathfrak{N}^n(D)$.*

(a) *If $(f_1, f_2, \dots, f_i, \dots, f_n) \in \mathfrak{N}_0^{n-1}(D)$ and f_i is locally univalent in D , then*

$$(2.3) \quad \frac{1}{\pi} \rho(z)^{-2} \iint_D \sum_{j \neq i} |S_{ij}(z, \zeta)|^2 d\xi d\eta \leq \pi \frac{\Gamma_i(f_i(z))}{\rho_i(f_i(z))^2} \leq 1, \quad z \in D.$$

(b) *If $(f_1, f_2, \dots, f_n) \in \mathfrak{N}_0^n(D)$, then*

(2.3')

$$\rho(z)^{-1} \left(\frac{1}{\pi} \iint_D \sum_{j=1}^n |S_{ij}(z, \zeta)|^2 d\xi d\eta \right)^{1/2} \leq 1 + \beta(z, D), \quad z \in D, 1 \leq i \leq n.$$

(c) *Let (f_1, f_2, \dots, f_n) be an n -tuple of meromorphic functions in D . If there is a constant K , such that for all $(z, w) \in D \times D$ and every $1 \leq i \leq n$, we have*

$$(2.3^*) \quad \rho(z)^{-1} \rho(w)^{-1} \sum_{j=1}^n |S_{ij}(z, \zeta)| \leq K < \infty,$$

then $(f_1, f_2, \dots, f_n) \in \mathfrak{N}_0^n(D)$.

PROOF. For a fixed $z \in D$ let $h_i(w) = (f_i(z) - w)^{-1}$ and, for $j \neq i$, $g_{ij}(\zeta) = h_i \circ f_j(\zeta) = (f_i(z) - f_j(\zeta))^{-1}$. Since $f_j(\zeta) \neq f_i(z)$ for every $\zeta \in D$, $g_{ij}(\zeta)$ is analytic in

D , and if $f_j(\zeta)$ is univalent in D , so is $g_{ij}(\zeta)$, and $S_{ij}(z, \zeta) = f'_i(z)g'_{ij}(\zeta)$. Hence, the univalence of $g_{ij}(\zeta)$ yields

$$\begin{aligned}
 (2.4) \quad & \iint_D \sum_{j \neq i} |S_{ij}(z, \zeta)|^2 d\xi d\eta = |f'_i(z)|^2 \sum_{j \neq i} \iint_D |g'_{ij}(\zeta)|^2 d\xi d\eta \\
 & = |f'_i(z)|^2 \sum_{j \neq i} \text{area } g_{ij}(D) = |f'_i(z)|^2 \text{area} \left[\bigcup_{j \neq i} h_i(D_j) \right] \\
 & \leq |f'_i(z)|^2 \text{area} [h_i(\hat{C} \setminus D_i)] = |f'_i(z)|^2 \iint_{\hat{C} \setminus D_i} |h'_i(\zeta)|^2 d\xi d\eta \\
 & = |f'_i(z)|^2 \iint_{\hat{C} \setminus D_i} \frac{d\xi d\eta}{|\zeta - f_i(z)|^4} = \pi^2 |f'_i(z)|^2 \Gamma_i(f_i(z)), \quad z \in D.
 \end{aligned}$$

But since $f_i: D \rightarrow D_i$ is locally univalent, we have

$$(2.5) \quad \rho(z) = \rho_i(f_i(z)) |f'_i(z)|, \quad z \in D.$$

Thus (2.4), (2.5) and (2.2) imply (2.3). Obviously there is an equality in (2.3) iff $\bigcup_{j=1}^n \bar{D}_j = \hat{C}$, and therefore (2.3) is sharp. (2.3') follows at once from (2.3) and (A.1).

Conversely, by (A.1'), if (2.3*) holds for some i , $1 \leq i \leq n$, then $f_i \in \mathfrak{N}_0(D)$; on the other hand, if $f_i(D) \cap f_j(D) \neq \emptyset$, one can find a pair $(z, \zeta) \in D \times D$ such that $f'_i(z)f'_j(\zeta) \neq 0$ but $f_i(z) = f_j(\zeta)$. This contradicts (2.3*) and completes the proof of the theorem.

From the reproduction property of $K_D(z, \bar{\zeta})$ and Cauchy-Schwarz' inequality it follows that every nonconstant $L^2(D)$ -integrable analytic function ϕ in D satisfies

$$(2.6) \quad |\phi(w)|^2 \leq K(w, \bar{w}) \|\phi\|_{L^2}^2 \leq \frac{1}{\pi} \rho(w)^2 \iint_D |\phi(\zeta)|^2 d\xi d\eta.$$

Hence, Theorem 1 implies

COROLLARY 1. *If $(f_1, f_2, \dots, f_n) \in \mathfrak{N}^n(D)$, $(f_1, \dots, f_i, \dots, f_n) \in \mathfrak{N}_0^{n-1}(D)$ and f_i locally univalent in D , then*

$$(2.7) \quad \rho_D(z)^{-2} \rho_D(w)^{-2} \sum_{j \neq i} |S_{ij}(z, w)|^2 \leq \pi \frac{\Gamma_i(f_i(z))}{\rho_i(f_i(z))^2} \leq 1, \quad (z, w) \in D \times D.$$

COROLLARY 2. *If $(f_1, f_2, \dots, f_n) \in \mathfrak{N}_0^n(D)$, then*

$$\begin{aligned}
 (2.8) \quad & \frac{\rho_D(z)^{-2}}{\pi} \iint_D |S_i(z, \zeta)|^2 d\xi d\eta + \rho_D(z)^{-2} \rho_D(w)^{-2} \sum_{j \neq i} |S_{ij}(z, w)|^2 \\
 & \leq (1 + \beta(z; D))^2 \leq 2^2, \quad (z, w) \in D \times D, 1 \leq i \leq n,
 \end{aligned}$$

and therefore

$$\begin{aligned}
 (2.8') \quad & \rho_D(z)^{-2} \rho_D(w)^{-2} \sum_{j=1}^n |S_{ij}(z, w)|^2 \leq (1 + \beta(z, D))^2 \leq 2^2, \\
 & (z, w) \in D \times D, 1 \leq i \leq n.
 \end{aligned}$$

REMARK. For $n = 2$ and $D =$ the unit disc U , (2.8) is Aharonov's inequality (cf. [2]);

(2.8*)

$$\frac{(1 - |z|^2)^2}{\pi} \iint_{|\xi| < 1} |S_f(z, \xi)|^2 d\xi d\eta + (1 - |z|^2)^2(1 - |w|^2)^2 |S_{f,g}(z, w)|^2 \leq 1,$$

$$|z| < 1, |w| < 1,$$

for all pairs $(f, g) \in \mathfrak{N}_0^2(U)$.

Inequality (2.6) with a second application of Cauchy-Schwarz' inequality yields

$$(2.6') \quad \rho_D(z)^{-1} \rho_D(w)^{-1} \sum_{j=1}^n |S_{ij}(z, w)|$$

$$\leq \sqrt{n} \rho_D(z)^{-1} \left(\frac{1}{\pi} \iint_D \sum_{j=1}^n |S_{ij}(z, \xi)|^2 d\xi d\eta \right)^{1/2}.$$

Hence we have

COROLLARY 3. Let f_1, f_2, \dots, f_n be meromorphic functions in D . Then the following are equivalent:

- (a) $(f_1, f_2, \dots, f_n) \in \mathfrak{N}_0^n(D)$;
- (b) $\sup_{z \in D} \rho_D(z)^{-1} \left(\frac{1}{\pi} \iint_D \sum_{j=1}^n |S_{ij}(z, \xi)|^2 d\xi d\eta \right)^{1/2} < \infty, 1 \leq i \leq n$;
- (c) $\sup_{z \in D} \sup_{w \in D} \rho_D(z)^{-1} \rho_D(w)^{-1} \sum_{j=1}^n |S_{ij}(z, w)| < \infty, 1 \leq i \leq n$.

3. The invariants $\psi_n(f_i, f_j; z)$. Applying the power series expansions of $S_{ij}(z, \xi)$, given in (1.1) and (1.2), and the orthogonality relations: $\iint_{|\xi| < 1} \xi^m \bar{\xi}^n d\xi d\eta = \pi \delta_{n,m} / (n + 1)$ in the unit disc U , we conclude from (2.3')

LEMMA 1. If $(f_1, f_2, \dots, f_n) \in \mathfrak{N}_0^n(U)$, then for every $1 \leq i \leq n$,

$$(3.1) \quad \sum_{m=2}^{\infty} (m - 1) \sum_{j=1}^n |\psi_m(f_i, f_j; 0)|^2 \leq 1 \quad (\psi_m(f_i, f_i; 0) = \psi_m(f_i, 0)).$$

The sequences $\{\psi_m(f_i, z)\}_{m=1}^{\infty}$ and $\{\psi_m(f_i, f_j; z)\}_{m=1}^{\infty}$ have the following generating functions (cf. Aharonov [1] and Aharonov and Lavie [3]):

$$(3.2) \quad \frac{f'_i(z)}{f_i(z) - f_i(\xi)} - \frac{1}{z - \xi} = \sum_{m=1}^{\infty} \psi_j(f_i, z)(\xi - z)^{m-1},$$

$$\frac{f'_i(z)}{f_i(z) - f_j(\xi)} = \sum_{m=1}^{\infty} \psi_m(f_i, f_j; z)(\xi - z)^{m-1}, \quad z \in D, |\xi - z| < d(z, \partial D),$$

from which it follows that for every $m \geq 2$, $\psi_m(f_i f_j; z)$ and $\psi_m(f_i, f_j; z) = \psi_m(f_i, z)$ are invariant in the sense:

$$(3.3) \quad \psi_m(g \circ f_i, g \circ f_j; z) = \psi_m(f_i, f_j; z) \quad \text{for every M\"obius transformation } g.$$

In [8] it has been proved:

LEMMA B. (a) Let $g \in \mathfrak{M}(D)$ and $f \in \mathfrak{M}(\tilde{D})$, $\tilde{D} = g(D)$. Then

$$(B.1) \quad \psi_m(f \circ g; z) = \sum_{k=1}^m B_{m,k}(g, z) \psi_k(f, g(z)) + \psi_m(g, z), \quad z \in D, m \geq 1,$$

where $B_{m,k}(g, z) = g'(z)A_{m-1,k-1}(g, z)$ and

$$(g(z) - g(\zeta))^k = \sum_{m=k}^{\infty} A_{m,k}(g, z)(\zeta - z)^m.$$

(b) If $g(z) = (az + b)/(cz + d)$, $ad - bc = 1$, then for $m \geq k \geq 2$

$$(B.1') \quad B_{m,k}(g, z) = \binom{m-2}{k-2} (-c)^{m-k} g'(z)^{(m+k)/2}, \quad \psi_m(g, z) = 0.$$

Similarly, one can easily prove:

LEMMA 2. If $g \in \mathfrak{M}(D)$ and $(f_1, f_2) \in \mathfrak{M}^2(\tilde{D})$, $\tilde{D} = g(D)$, then

$$(3.4) \quad \psi_m(f_1 \circ g, f_2 \circ g; z) = \sum_{k=2}^m B_{m,k}(g, z) \psi_k(f_1, f_2; g(z)), \quad z \in D, m \geq 1,$$

and if $g(z) = (az + b)/(cz + d)$, $ad - bc = 1$, then

$$(3.4') \quad \psi_m(f_1 \circ g, f_2 \circ g; z) = \sum_{k=2}^m \binom{m-2}{k-2} (-c)^{m-k} g'(z)^{(m+k)/2} \psi_k(f_1, f_2; g(z)).$$

Lemmas 1 and 2 imply (cf. [7 and 8])

THEOREM 2. (a) If $(f_1, f_2, \dots, f_n) \in \mathfrak{M}_0^n(D)$, then

$$(3.5) \quad \sum_{m=2}^{\infty} (m-1) d(z, \partial D)^{2m} \sum_{j=1}^n |\psi_m(f_i, f_j; z)|^2 \leq 1, \quad z \in D, 1 \leq i \leq n.$$

(b) If $(f_1, f_2, \dots, f_n) \in \mathfrak{M}_0^n(U)$, then

$$(3.6) \quad \sum_{m=2}^{\infty} (m-1) \sum_{j=1}^n \left| \sum_{l=2}^m \binom{m-2}{l-2} (-\bar{\zeta})^{m-l} (1 - |\zeta|^2)^l \psi_l(f_i, f_j; \zeta) \right|^2 \leq 1,$$

$|\zeta| < 1, 1 \leq i \leq n,$

$$(3.7) \quad (1 - |z|^2)^m |\psi_m(f_i, f_j; z)| \leq p_{m-2}(|z|) = \sum_{k=2}^m \binom{m-2}{k-2} \frac{|z|^{m-k}}{\sqrt{k-1}},$$

$|z| < 1, 1 \leq i, j \leq n,$

and

$$(3.7') \quad (1 - |z|^2)^m \sum_{j=1}^n |\psi_m(f_i, f_j; z)| \leq \sqrt{n} p_{m-2}(|z|), \quad |z| < 1, 1 \leq i \leq n.$$

PROOF. (a) For any $z \in D$ let $g(\zeta) = r\zeta + z$, where $r = d(z, \partial D)$. Then by (3.4') $\psi_m(f_i \circ g, f_j \circ g; \zeta) = r^m \psi_m(f_i, f_j; r\zeta + z)$ for every $\zeta \in U$, and $(f_1 \circ g, f_2 \circ g, \dots, f_n \circ g)|_U \in \mathfrak{M}_0^n(U)$. Thus (3.1) implies (3.5).

(b) Similarly (3.1) implies (3.6), using the Möbius self-mapping

$$g(z) = (z + \zeta) / (1 + \bar{\zeta}z)$$

of U and (3.4'). For the proof of (3.7) let $g(\zeta) = (\zeta - z)/(1 - \bar{z}\zeta)$. Then by (B.1') and (3.4'),

$$(3.8) \quad (1 - |z|^2)^m \psi_m(f_i, f_j; z) = \sum_{k=2}^m \binom{m-2}{k-2} \psi_k(f_i \circ g^{-1}, f_j \circ g^{-1}; 0) \bar{z}^{m-k}$$

(since $g(z) = 0$).

Notice also that (3.1) yields

$$(3.9) \quad \sum_{j=1}^n |\psi_m(f_i \circ g^{-1}, f_j \circ g^{-1}; 0)|^2 \leq \frac{1}{m-1}, \quad m \geq 2, 1 \leq i, j \leq n,$$

and in particular $|\psi_m(f_i \circ g^{-1}, f_j \circ g^{-1}; 0)| \leq 1/\sqrt{m-1}$. This implies (3.7) at once, and also the inequality

$$\begin{aligned} (1 - |z|^2)^m \left| \sum_{j=1}^n \psi_m(f_i, f_j; z) e^{i\theta_j} \right| \\ \leq \sum_{k=2}^m \binom{m-2}{k-2} |z|^{m-k} \sum_{j=1}^n |\psi_k(f_i \circ g^{-1}, f_j \circ g^{-1}; 0)| \\ \leq \sqrt{n} p_{m-2}(|z|), \end{aligned}$$

for every choice of $\theta_1, \theta_2, \dots, \theta_n$. Now take θ_j such that

$$\psi_m(f_i, f_j; z) e^{i\theta_j} = |\psi_m(f_i, f_j; z)|$$

and the proof of (3.7') is complete. Q.E.D.

4. The invariants $\phi_n(f_i, f_j; z)$. In this section we consider a second sequence of invariants

$$(4.1) \quad \phi_n(f_1, f_2, z) = \frac{3!}{(n+1)!} \phi_2^{(n-2)}(f_1, f_2; z), \quad n \geq 2, (f_1, f_2) \in \mathfrak{N}_0^2(D),$$

where

$$\phi_2(f_1, f_2; z) = \psi_2(f_1, f_2; z) = \frac{f_1'(z)f_2'(z)}{(f_1(z) - f_2(z))^2} = S_{f_1, f_2}(z, z),$$

and formally define

$$(4.2) \quad T_{f_1, f_2}(z, \zeta) = \sum_{n=2}^{\infty} (n-1) \phi_n(f_1, f_2; z) (\zeta - z)^{n-2},$$

$$z \in D, |\zeta - z| < d(z, \partial D).$$

(Cf. the definitions of $\phi_n(f, z)$ and $T_f(z, \zeta)$ for a single function $f \in \mathfrak{N}(D)$ in [10].)

THEOREM 3. (a) *Let D be a Jordan domain the boundary $C = \partial D$ of which is a quasicircle. Then there is constant $K = K(C)$, depending only on C , such that*

$$(4.3) \quad \rho_D = (z)^{-1} \left(\frac{1}{\pi} \iint_D |T_{f_1, f_2}(z, \zeta)|^2 d\xi d\eta \right)^{1/2} \leq K(C)$$

for every pair $(f_1, f_2) \in \mathfrak{N}_0^2(D)$.

(b) *If $C = \partial U$ is the unit circle, then $K(C) \leq 3$.*

PROOF. By Corollary 1, if $(f_1, f_2) \in \mathfrak{N}^2(D)$, f_1 locally univalent and $f_2 \in \mathfrak{N}_0(D)$, then

$$\sup_{z \in D} \rho_D(z)^{-2} |\phi_2(f_1, f_2; z)| \leq \sup_{z, w \in D} \rho_D(z)^{-1} \rho_D(w)^{-1} |S_{f_1, f_2}(z, w)| \leq 1.$$

Hence, by Theorem 1 in [10], if $C = \partial D$ is a quasicircle, then

$$\begin{aligned} \rho_D(z)^{-1} \left(\frac{1}{\pi} \iint_D |T_{f_1, f_2}(z, \zeta)|^2 d\xi d\eta \right)^{1/2} \\ \leq K(C) \sup_{z \in D} \rho_D(z)^{-2} |\phi_2(f_1, f_2; z)| \leq K(C) \end{aligned}$$

for some constant $K(C)$ depending only on C , and $K(C) \leq 3$ if $C = \partial U$ is the unit circle. Q.E.D.

From (4.2), (4.3) and Parseval's identity we conclude

LEMMA 3. *If $(f_1, f_2) \in \mathfrak{N}_0^2(U)$, then*

$$(4.4) \quad \sum_{n=2}^{\infty} (n-1) |\phi_n(f_1, f_2; 0)|^2 \leq K(\partial U)^2 \leq 9.$$

Computations similar to those appearing in [10] yield

LEMMA 4. *For every pair $(f_1, f_2) \in \mathfrak{N}^2(D)$ and $g(z) = (az + b)/(cz + d)$, $ad - bc = 1$,*

$$(4.5) \quad T_{f_1 \circ g, f_2 \circ g}(z, \zeta) = T_{f_1, f_2}(g(z), g(\zeta))g'(z)g'(\zeta)$$

and

$$(4.6) \quad \phi_n(f_1 \circ g, f_2 \circ g; z) = \sum_{l=2}^n \binom{n-2}{l-2} (-c)^{n-l} g'(z)^{(n+l)/2} \phi_l(f_1, f_2; g(z)).$$

Lemmas 3 and 4 imply the following analogue for Theorem 2.

THEOREM 4. (a) *If $(f_1, f_2) \in \mathfrak{N}_0^2(D)$, then*

$$(4.7) \quad \sum_{n=2}^{\infty} (n-1) d(z, \partial D)^{2n} |\phi_n(f_1, f_2; z)|^2 \leq 9, \quad z \in D.$$

(b) *If $(f_1, f_2) \in \mathfrak{N}_0^2(U)$, then*

$$(4.8) \quad \sum_{n=2}^{\infty} (n-1) \left| \sum_{l=2}^n \binom{n-2}{l-2} (-\bar{\zeta})^{n-l} (1 - |\zeta|^2)^l \phi_l(f_1, f_2; \zeta) \right|^2 \leq 0, \quad |\zeta| < 1,$$

and

$$(4.9) \quad (1 - |z|^2)^n |\phi_n(f_1, f_2; z)| \leq 3p_{n-2}(|z|)$$

where $p_{n-2}(|z|)$ is the real polynomial defined in (3.7).

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