

ON THE SOBCZYK-HAMMER DECOMPOSITION OF ADDITIVE SET FUNCTIONS

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ABSTRACT. It is observed that continuity for charges is equivalent to the absence of two-valued minorants. This characterization forms the basis of a new short proof within a functional-analytic context of a decomposition theorem by A. Sobczyk and P. C. Hammer [5] for charges on a field \mathfrak{A} into a continuous part and a part which can be written as a sum of at most two-valued charges on \mathfrak{A} . A counterexample shows that in general the decomposition of a charge into a nonatomic part and a part which has no nonnull nonatomic minorant is not unique.

1. Definitions. Let \mathfrak{A} be a field of subsets of a set Ω . A *charge* μ on \mathfrak{A} is a real-valued nonnegative finitely additive function defined on \mathfrak{A} . A *measure* is a countably additive charge whose domain is a σ -field of subsets of a set. A set $A \in \mathfrak{A}$ will be called an *atom* for μ iff $\mu(A) > 0$ and for every set $E \in \mathfrak{A}$, $E \subset A$, either $\mu(E) = 0$ or $\mu(E) = \mu(A)$. μ is *nonatomic* iff there are no atoms for μ . Finally, a charge μ on a field \mathfrak{A} (in Ω) is said to be *continuous* iff, given $\varepsilon > 0$, there exists a partition $P = \{B_1, \dots, B_n\}$ of Ω into a finite number of pairwise disjoint members of \mathfrak{A} such that $\mu(B_i) < \varepsilon$ for every i . Let \mathfrak{P} denote the family of all such partitions, and $|P(\mu)|$ the maximum of $\mu(B_i)$ over all parts B_i of $P \in \mathfrak{P}$.

2. Main result. It is well known that a measure λ on a σ -field \mathfrak{C} can be decomposed uniquely into $\lambda = \lambda_0 + \lambda'$ where λ' is a nonatomic measure on \mathfrak{C} and λ_0 a completely atomic measure on \mathfrak{C} , i.e., λ_0 has the form $\lambda_0 = \sum_{i=1}^{\infty} \lambda_i$ with at most two-valued measures λ_i on \mathfrak{C} ($i \in \mathbb{N}$).

A. Sobczyk and P. C. Hammer have proved an analogous decomposition theorem for charges on fields of subsets of a set:

DECOMPOSITION THEOREM (SOBCZYK-HAMMER [5, THEOREM 4.1]). *Let μ be a charge on a field \mathfrak{A} (in a set Ω). Then μ can be decomposed uniquely into $\mu = \mu_0 + \mu'$ with a continuous charge μ' on \mathfrak{A} and a charge μ_0 on \mathfrak{A} which has the form $\mu_0 = \sum_{i=1}^{\infty} \mu_i$ with at most two-valued charges μ_i on \mathfrak{A} ($i \in \mathbb{N}$).*

The following lemma yields a reformulation of this theorem, for which a short functional-analytic proof will be given. The Krein-Milman representation theorem will serve as a link to the classical Lebesgue decomposition theorem.

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LEMMA. Let ν be a charge on a field \mathfrak{A} . ν is continuous iff ν admits no two-valued minorant, i.e., ν has the property (*) $\nu \geq \eta$ implies $\eta = 0$ for any at most two-valued charge η on \mathfrak{A} .

PROOF. A canonical indirect argument shows that a continuous charge on \mathfrak{A} also has the property (*). Now suppose that ν is not continuous. Then by Lemmas 4.1 and 4.2 of [5] there exists a two-valued charge ν^* on \mathfrak{A} with $\nu \geq \nu^*$: Firstly, $\alpha = \inf_{P \in \mathfrak{P}} |P(\nu)| > 0$ implies the existence of a set $\Omega_0 \in \mathfrak{A}$ with (i) $\alpha \leq \nu(\Omega_0) < 2\alpha$, and such that every partition $P \in \mathfrak{P}$ contains some part B_i with (ii) $\alpha \leq \nu(B_i \Omega_0)$. Notice that for any $\varepsilon > 0$ there is a partition $P_0 \in \mathfrak{P}$ with $|P_0(\nu)| < \alpha + \varepsilon$. Choose $\varepsilon < \alpha$. Then by $\alpha = \inf_{P \in \mathfrak{P}} |P(\nu)| > 0$ the family $\{C_1, \dots, C_k\}$ of members from P_0 with $\alpha \leq \nu(C_i) < 2\alpha$ for every i is nonempty and has a member Ω_0 such that for every partition $P = \{B_1, \dots, B_n\} \in \mathfrak{P}$ there is an index i with $\alpha \leq \nu(B_i \Omega_0)$. Secondly, let ν^* be defined on \mathfrak{A} by $\nu^*(A) = \alpha$, resp. 0, if $\nu(A \Omega_0) \geq \alpha$, resp. $\nu(A \Omega_0) < \alpha$, for every $A \in \mathfrak{A}$. Then ν^* is the desired two-valued charge: Consider disjoint sets $A, B \in \mathfrak{A}$ with $\nu((A + B)\Omega_0) \geq \alpha$. Then property (i) implies $\nu((\Omega_0 \setminus A \Omega_0) \setminus B \Omega_0) < \alpha$. Thus by (ii) either $\nu(A \Omega_0) \geq \alpha$ or $\nu(B \Omega_0) \geq \alpha$ (but not both, by (i) again). Hence $\nu^*(A + B) = \alpha = \nu^*(A) + \nu^*(B)$, which finishes the proof.

THEOREM. Let μ be a charge on a field \mathfrak{A} of subsets of a set Ω . Then μ is decomposable uniquely into $\mu = \mu_0 + \mu'$ where μ' is a charge on \mathfrak{A} with property (*) and μ_0 is a charge on \mathfrak{A} of the form $\mu_0 = \sum_{i=1}^{\infty} \mu_i$ with at most two-valued charges μ_i on \mathfrak{A} ($i \in \mathbf{N}$).

PROOF. Existence. Let μ be a probability charge on \mathfrak{A} . Let X denote the set of all probability charges on \mathfrak{A} and X_e the set of extreme points of X . Then by the theorem of Krein-Milman [3, p. 6] there exists a regular probability measure ρ on the Borel sets of X with $\rho(X_e) = 1$ and the property $\mu(A) = \int_{X_e} \nu(A) \rho(d\nu)$ for every $A \in \mathfrak{A}$. Notice that X is a convex and by the theorem of Tychonoff compact subset of the locally convex space $E = \text{ba}(\Omega, \mathfrak{A})$ of all bounded additive set functions on \mathfrak{A} under the weak* topology and that the (by [2, V.8.2] nonempty) set X_e characterized by the set of the $\{0, 1\}$ -valued charges on \mathfrak{A} [1, p. 245] is weak* closed. The measure ρ has a unique decomposition into a convex combination $\rho = \alpha \rho_1 + (1 - \alpha) \rho_2$ of a discrete probability measure ρ_1 and a probability measure ρ_2 with $\rho_2(\{\nu\}) = 0$ for all $\nu \in X$, both on the Borel sets of X [2, III.4.14]. Thus ρ_1 can be represented by $\rho_1 = \sum_{i=1}^{\infty} \alpha_i \varepsilon_{\nu_i}$ where ε_{ν_j} denotes the Dirac measure on the Borel sets of X at $\nu_j \in X_e$ and where $\alpha_j \geq 0$ with $\sum_{i=1}^{\infty} \alpha_i = 1$ ($j \in \mathbf{N}$). Further, the probability charge ν' on \mathfrak{A} defined by $\nu'(A) = \int_{X_e} \nu(A) \rho_2(d\nu)$ for every $A \in \mathfrak{A}$ has the property (*):

Suppose, there is a two-valued charge $\tilde{\nu}$ on \mathfrak{A} with $\nu' \geq \tilde{\nu}$. Define $\tilde{\nu} = \tilde{\nu}/\tilde{\nu}(\Omega)$, $\Delta = \{A \in \mathfrak{A} \mid \tilde{\nu}(A) = 1\}$ and $U_A = \{\nu \in X_e \mid \nu(A) = 1\}$ for every $A \in \mathfrak{A}$. U_A is weak* closed, and for every $A \in \Delta$ the following chain holds:

$$\begin{aligned} \nu'(A) &= \int_{X_e} \nu(A) \rho_2(d\nu) = \int_{U_A} \nu(A) \rho_2(d\nu) + \int_{X_e \setminus U_A} \nu(A) \rho_2(d\nu) \\ &= \rho_2(U_A) \geq \tilde{\nu}(A) > 0 \end{aligned}$$

and therefore $\inf_{\Delta} \rho_2(U_A) > 0$. Because ρ_2 is a bounded regular measure on the Borel sets of a compact space, $\rho_2(\bigcap_{\Delta} U_A) = \inf_{\Delta} \rho_2(U_A)$. (Notice that for any $\varepsilon > 0$ there is a weak* compact subset K of $\bigcup_{\Delta} V_A$ with $V_A = X \setminus U_A$, $A \in \Delta$, such that $\rho_2(K) > \rho_2(\bigcup_{\Delta} V_A) - \varepsilon$. Thus, there exists a finite family $\{A_1, \dots, A_n\} \subset \Delta$ with $K \subset \bigcup_{i=1}^n V_{A_i}$. Now, define $A_0 = \bigcap_{i=1}^n A_i$. Then, $A_0 \in \Delta$ and $\bigcup_{i=1}^n V_{A_i} \subset V_{A_0}$ [the first because of $\bigcap_{j=1}^m U_{B_j} = U_{\bigcap_{j=1}^m B_j}$ for all $m \in \mathbb{N}$, $B_1, \dots, B_m \in \mathfrak{A}$]. Therefore $\rho_2(V_{A_0}) \geq \rho_2(K) > \rho_2(\bigcup_{\Delta} V_A) - \varepsilon$. Hence $\rho_2(\bigcup_{\Delta} U_A^c) = \sup_{\Delta} \rho_2(U_A^c)$.) Finally, $\bigcap_{\Delta} U_A = \{\tilde{v}\}$ holds, which implies $\rho_2(\{\tilde{v}\}) > 0$ in contradiction to the property of ρ_2 . Then $\mu(A) = \alpha \sum_{i=1}^{\infty} \alpha_i \nu_i(A) + (1 - \alpha) \nu'(A)$ for every $A \in \mathfrak{A}$. Thus the existence of a Sobczyk-Hammer decomposition for μ is established.

Uniqueness. Let μ be an element of X . Then there exists a unique regular probability measure ρ on the Borel sets of X with $\rho(X_e) = 1$ which represents μ (in the sense of [3, p. 2]). Suppose $\tilde{\rho}$ is another measure with the same property. Then $\rho(U_A) = \tilde{\rho}(U_A)$ for every $A \in \mathfrak{A}$. Consequently, ρ and $\tilde{\rho}$ coincide on arbitrary unions of sets from $\nabla = \{U_A \mid A \in \mathfrak{A}\}$. (Notice that a similar argument as used in the proof for existence shows $\rho(\bigcap_{\phi} Z_A) = \inf_{\phi} \rho(Z_A)$ with a subfamily ϕ of \mathfrak{A} which is closed with respect to finite unions, and $Z_A = \{\nu \in X_e \mid \nu(A) = 0\}$, $A \in \phi$. Paying attention to $\rho(X_e) = \tilde{\rho}(X_e) = 1$, this identity implies the preceding coincidence statement.) Because X_e is compact with respect to the relative weak* topology and ∇ is a base for this topology ρ and $\tilde{\rho}$ coincide on the Borel sets of X_e and therefore also on the Borel sets of X .

It follows that the mapping,

$$\begin{aligned} X &\rightarrow M_1^+(X_e), \\ \mu &\rightarrow \rho_{\mu}, \end{aligned}$$

where $M_1^+(X_e)$ is the set of all regular probability measures on the Borel sets of X which are concentrated at X_e and where ρ_{μ} denotes the regular probability measure which represents μ , is an affine homeomorphism with (i) ρ_{μ} is discrete iff μ has the form $\mu = \sum_{i=1}^{\infty} \mu_i$ with at most two-valued charges μ_j ($j \in \mathbb{N}$) and (ii) ρ_{μ} is continuous, i.e. vanishing at every singleton, iff μ is continuous. This completes the proof for uniqueness.

REMARKS. (1) To prove the existence of a Sobczyk-Hammer decomposition for a charge μ on a field \mathfrak{A} of subsets of a set Ω it also suffices to show that the system $\mathfrak{S} = \{\sum_{i=1}^{\infty} \mu_i \mid \mu_j \text{ is an at most two-valued charge on } \mathfrak{A} \text{ for every } j \in \mathbb{N} \text{ with } \sum_{i=1}^{\infty} \mu_i \leq \mu\}$ is inductively ordered with respect to the usual partial ordering \leq on $\text{ba}(\Omega, \mathfrak{A})$, because in that case Zorn's lemma implies the existence of a maximal element $\bar{\mu}$ in \mathfrak{S} . Thus μ can be decomposed into $\mu = \bar{\mu} + \mu'$ where $\mu' = \mu - \bar{\mu} \geq 0$. The charge μ' has the property (*).

(2) By the same technique, the following decomposition property can be derived: Let μ be a charge on a field \mathfrak{A} of subsets of a set Ω . Then μ can be decomposed into $\mu = \mu_0 + \mu'$ where μ_0 is a nonatomic charge on \mathfrak{A} and μ' is a charge on \mathfrak{A} which has no nonnull nonatomic minorant, i.e., μ' has the property (*) $\mu' \geq \bar{\mu}_0$ implies $\bar{\mu}_0 = 0$ for every nonatomic charge $\bar{\mu}_0$ on \mathfrak{A} .

PROOF. Consider the system $\mathfrak{S} = \{\bar{\mu} \mid \bar{\mu} \text{ is a nonatomic charge on } \mathfrak{A} \text{ with } \bar{\mu} \leq \mu\}$. \mathfrak{S} is inductively ordered with respect to the natural partial ordering \leq on $\text{ba}(\Omega, \mathfrak{A})$: Let \mathfrak{S}_0 be a subset of \mathfrak{S} which is linearly ordered. Then the charge μ_0 defined by $\mu_0(A) = \sup\{\bar{\mu}(A) \mid \bar{\mu} \in \mathfrak{S}_0\}$ for every $A \in \mathfrak{A}$ is nonatomic and therefore an upper bound for \mathfrak{S}_0 in \mathfrak{S} . The nonatomicity of μ_0 can be shown by a more canonical indirect argument.

A counterexample shows that in general the preceding decomposition of a charge is not unique:

Define $\Omega = [0, 1]$, $\mathfrak{C} = \{]a, b[\mid]a, b[\text{ interval with }]a, b[\subset]\frac{1}{4}, \frac{3}{4}[\}$, let \mathfrak{A} denote the field of subsets of Ω which is generated by \mathfrak{C} and let $\mu = \lambda \mid \mathfrak{A}$ be the restriction of the Lebesgue measure λ on \mathfrak{A} . Then μ is a nonatomic charge on \mathfrak{A} . There are two decompositions for μ of the above type: First, choose $\mu_0 = \mu$, μ' vanishing at every $A \in \mathfrak{A}$. Secondly, choose μ' according to $\mu'(B) = \lambda(B \cap A)$ for every $B \in \mathfrak{A}$ where $A = [0, \frac{1}{4}[\cup]\frac{3}{4}, 1]$ and $\mu_0 = \mu - \mu'$. Then the following statements hold: (i) μ' is a charge on \mathfrak{A} with $\mu' \leq \mu$, (ii) μ' is two-valued and therefore has the property (*) because of the fact that $A \subset B$ or $B \subset]\frac{1}{4}, \frac{3}{4}[$ and thus $\mu'(B) = \frac{1}{2}$ or $\mu'(B) = 0$ for every $B \in \mathfrak{A}$, and (iii) μ_0 is nonatomic.

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