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ON THE SOBCZYK-HAMMER DECOMPOSITION OF ADDITIVE SET FUNCTIONS

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ABSTRACT. It is observed that continuity for charges is equivalent to the absence of two-valued minorants. This characterization forms the basis of a new short proof within a functional-analytic context of a decomposition theorem by A. Sobczyk and P. C. Hammer [5] for charges on a field $\mathfrak{A}$ into a continuous part and a part which can be written as a sum of at most two-valued charges on $\mathfrak{A}$. A counterexample shows that in general the decomposition of a charge into a nonatomic part and a part which has no nonnull nonatomic minorant is not unique.

1. Definitions. Let $\mathfrak{A}$ be a field of subsets of a set Ω. A *charge* μ on $\mathfrak{A}$ is a real-valued nonnegative finitely additive function defined on $\mathfrak{A}$. A *measure* is a countably additive charge whose domain is a σ-field of subsets of a set. A set $A \in \mathfrak{A}$ will be called an *atom* for μ iff $\mu(A) > 0$ and for every set $E \in \mathfrak{A}$, $E \subset A$, either $\mu(E) = 0$ or $\mu(E) = \mu(A)$. μ is *nonatomic* iff there are no atoms for μ. Finally, a charge μ on a field $\mathfrak{A}$ (in Ω) is said to be *continuous* iff, given $\varepsilon > 0$, there exists a partition $P = \{B_1, \dots, B_n\}$ of Ω into a finite number of pairwise disjoint members of $\mathfrak{A}$ such that $\mu(B_i) < \varepsilon$ for every i. Let $\mathfrak{P}$ denote the family of all such partitions, and $|P(\mu)|$ the maximum of $\mu(B_i)$ over all parts B_i of $P \in \mathfrak{P}$.

2. Main result. It is well known that a measure λ on a σ-field $\mathfrak{S}$ can be decomposed uniquely into $\lambda = \lambda_0 + \lambda'$ where λ' is a nonatomic measure on $\mathfrak{S}$ and λ_0 a completely atomic measure on $\mathfrak{S}$, i.e., λ_0 has the form $\lambda_0 = \sum_{i=1}^{\infty} \lambda_i$ with at most two-valued measures λ_i on $\mathfrak{S}$ ($i \in \mathbf{N}$).

A. Sobczyk and P. C. Hammer have proved an analogous decomposition theorem for charges on fields of subsets of a set:

DECOMPOSITION THEOREM (SOBCZYK-HAMMER [5, THEOREM 4.1]). *Let μ be a charge on a field $\mathfrak{A}$ (in a set Ω). Then μ can be decomposed uniquely into $\mu = \mu_0 + \mu'$ with a continuous charge μ' on $\mathfrak{A}$ and a charge μ_0 on $\mathfrak{A}$ which has the form $\mu_0 = \sum_{i=1}^{\infty} \mu_i$ with at most two-valued charges μ_i on $\mathfrak{A}$ ($i \in \mathbf{N}$).*

The following lemma yields a reformulation of this theorem, for which a short functional-analytic proof will be given. The Krein-Milman representation theorem will serve as a link to the classical Lebesgue decomposition theorem.

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LEMMA. *Let ν be a charge on a field $\mathfrak{A}$. ν is continuous iff ν admits no two-valued minorant, i.e., ν has the property* $(*)$ $\nu \geqslant \eta$ *implies* $\eta = 0$ *for any at most two-valued charge η on $\mathfrak{A}$.*

PROOF. A canonical indirect argument shows that a continuous charge on $\mathfrak{A}$ also has the property $(*)$. Now suppose that ν is not continuous. Then by Lemmas 4.1 and 4.2 of [**5**] there exists a two-valued charge ν^* on $\mathfrak{A}$ with $\nu \geqslant \nu^*$: Firstly, $\alpha = \inf_{P \in \mathfrak{P}} |P(\nu)| > 0$ implies the existence of a set $\Omega_0 \in \mathfrak{A}$ with (i) $\alpha \leqslant \nu(\Omega_0) < 2\alpha$, and such that every partition $P \in \mathfrak{P}$ contains some part B_i with (ii) $\alpha \leqslant \nu(B_i\Omega_0)$. Notice that for any $\varepsilon > 0$ there is a partition $P_0 \in \mathfrak{P}$ with $|P_0(\nu)| < \alpha + \varepsilon$. Choose $\varepsilon < \alpha$. Then by $\alpha = \inf_{P \in \mathfrak{P}} |P(\nu)| > 0$ the family $\{C_1, \ldots, C_k\}$ of members from P_0 with $\alpha \leqslant \nu(C_i) < 2\alpha$ for every i is nonempty and has a member Ω_0 such that for every partition $P = \{B_1, \ldots, B_n\} \in \mathfrak{P}$ there is an index i with $\alpha \leqslant \nu(B_i\Omega_0)$. Secondly, let ν^* be defined on $\mathfrak{A}$ by $\nu^*(A) = \alpha$, resp. 0, if $\nu(A\Omega_0) \geqslant \alpha$, resp. $\nu(A\Omega_0) < \alpha$, for every $A \in \mathfrak{A}$. Then ν^* is the desired two-valued charge: Consider disjoint sets A, $B \in \mathfrak{A}$ with $\nu((A + B)\Omega_0) \geqslant \alpha$. Then property (i) implies $\nu((\Omega_0 \setminus A\Omega_0) \setminus B\Omega_0) < \alpha$. Thus by (ii) either $\nu(A\Omega_0) \geqslant \alpha$ or $\nu(B\Omega_0) \geqslant \alpha$ (but not both, by (i) again). Hence $\nu^*(A + B) = \alpha = \nu^*(A) + \nu^*(B)$, which finishes the proof.

THEOREM. *Let μ be a charge on a field $\mathfrak{A}$ of subsets of a set Ω. Then μ is decomposable uniquely into $\mu = \mu_0 + \mu'$ where μ' is a charge on $\mathfrak{A}$ with property* $(*)$ *and μ_0 is a charge on $\mathfrak{A}$ of the form $\mu_0 = \sum_{i=1}^{\infty} \mu_i$ with at most two-valued charges μ_i on $\mathfrak{A}$ $(i \in \mathbf{N})$.*

PROOF. *Existence*. Let μ be a probability charge on $\mathfrak{A}$. Let X denote the set of all probability charges on $\mathfrak{A}$ and X_e the set of extreme points of X. Then by the theorem of Krein-Milman [**3**, p. 6] there exists a regular probability measure ρ on the Borel sets of X with $\rho(X_e) = 1$ and the property $\mu(A) = \int_{X_e} \nu(A)\rho(d\nu)$ for every $A \in \mathfrak{A}$. Notice that X is a convex and by the theorem of Tychonoff compact subset of the locally convex space $E = \mathrm{ba}(\Omega, \mathfrak{A})$ of all bounded additive set functions on $\mathfrak{A}$ under the weak* topology and that the (by [**2**, V.8.2] nonempty) set X_e characterized by the set of the $\{0, 1\}$-valued charges on $\mathfrak{A}$ [**1**, p. 245] is weak* closed. The measure ρ has a unique decomposition into a convex combination $\rho = \alpha\rho_1 + (1 - \alpha)\rho_2$ of a discrete probability measure ρ_1 and a probability measure ρ_2 with $\rho_2(\{\nu\}) = 0$ for all $\nu \in X$, both on the Borel sets of X [**2**, III.4.14]. Thus ρ_1 can be represented by $\rho_1 = \sum_{i=1}^{\infty} \alpha_i \varepsilon_{\nu_i}$ where ε_{ν_j} denotes the Dirac measure on the Borel sets of X at $\nu_j \in X_e$ and where $\alpha_j \geqslant 0$ with $\sum_{i=1}^{\infty} \alpha_i = 1$ $(j \in \mathbf{N})$. Further, the probability charge ν' on $\mathfrak{A}$ defined by $\nu'(A) = \int_{X_e} \nu(A)\rho_2(d\nu)$ for every $A \in \mathfrak{A}$ has the property $(*)$:

Suppose, there is a two-valued charge $\tilde{\nu}$ on $\mathfrak{A}$ with $\nu' \geqslant \tilde{\nu}$. Define $\tilde{\tilde{\nu}} = \tilde{\nu}/\tilde{\nu}(\Omega)$, $\Delta = \{A \in \mathfrak{A} \mid \tilde{\tilde{\nu}}(A) = 1\}$ and $U_A = \{\nu \in X_e \mid \nu(A) = 1\}$ for every $A \in \mathfrak{A}$. U_A is weak* closed, and for every $A \in \Delta$ the following chain holds:

$$\begin{aligned} \nu'(A) &= \int_{X_e} \nu(A)\rho_2(d\nu) = \int_{U_A} \nu(A)\rho_2(d\nu) + \int_{X_e \setminus U_A} \nu(A)\rho_2(d\nu) \\ &= \rho_2(U_A) \geqslant \tilde{\nu}(A) > 0 \end{aligned}$$

and therefore $\inf_\Delta \rho_2(U_A) > 0$. Because ρ_2 is a bounded regular measure on the Borel sets of a compact space, $\rho_2(\bigcap_\Delta U_A) = \inf_\Delta \rho_2(U_A)$. (Notice that for any $\varepsilon > 0$ there is a weak* compact subset K of $\bigcup_\Delta V_A$ with $V_A = X \setminus U_A$, $A \in \Delta$, such that $\rho_2(K) > \rho_2(\bigcup_\Delta V_A) - \varepsilon$. Thus, there exists a finite family $\{A_1, \ldots, A_n\} \subset \Delta$ with $K \subset \bigcup_{i=1}^n V_{A_i}$. Now, define $A_0 = \bigcap_{i=1}^n A_i$. Then, $A_0 \in \Delta$ and $\bigcup_{i=1}^n V_{A_i} \subset V_{A_0}$ [the first because of $\bigcap_{j=1}^m U_{B_j} = U_{\bigcap_{j=1}^m B_j}$ for all $m \in \mathbf{N}$, $B_1, \ldots, B_m \in \mathfrak{A}$]. Therefore $\rho_2(V_{A_0}) \geqslant \rho_2(K) > \rho_2(\bigcup_\Delta V_A) - \varepsilon$. Hence $\rho_2(\bigcup_\Delta U_A^c) = \sup_\Delta \rho_2(U_A^c)$.) Finally, $\bigcap_\Delta U_A = \{\tilde{\bar{\nu}}\}$ holds, which implies $\rho_2(\{\tilde{\bar{\nu}}\}) > 0$ in contradiction to the property of ρ_2. Then $\mu(A) = \alpha \sum_{i=1}^\infty \alpha_i \nu_i(A) + (1-\alpha)\nu'(A)$ for every $A \in \mathfrak{A}$. Thus the existence of a Sobczyk-Hammer decomposition for μ is established.

Uniqueness. Let μ be an element of X. Then there exists a unique regular probability measure ρ on the Borel sets of X with $\rho(X_e) = 1$ which represents μ (in the sense of [**3**, p. 2]). Suppose $\tilde{\rho}$ is another measure with the same property. Then $\rho(U_A) = \tilde{\rho}(U_A)$ for every $A \in \mathfrak{A}$. Consequently, ρ and $\tilde{\rho}$ coincide on arbitrary unions of sets from $\nabla = \{U_A \mid A \in \mathfrak{A}\}$. (Notice that a similar argument as used in the proof for existence shows $\rho(\bigcap_\phi Z_A) = \inf_\phi \rho(Z_A)$ with a subfamily ϕ of $\mathfrak{A}$ which is closed with respect to finite unions, and $Z_A = \{\nu \in X_e \mid \nu(A) = 0\}$, $A \in \phi$. Paying attention to $\rho(X_e) = \tilde{\rho}(X_e) = 1$, this identity implies the preceding coincidence statement.) Because X_e is compact with respect to the relative weak* topology and ∇ is a base for this topology ρ and $\tilde{\rho}$ coincide on the Borel sets of X_e and therefore also on the Borel sets of X.

It follows that the mapping,

$$X \to M_1^+(X_e),$$
$$\mu \to \rho_\mu,$$

where $M_1^+(X_e)$ is the set of all regular probability measures on the Borel sets of X which are concentrated at X_e and where ρ_μ denotes the regular probability measure which represents μ, is an affine homeomorphism with (i) ρ_μ is discrete iff μ has the form $\mu = \sum_{i=1}^\infty \mu_i$ with at most two-valued charges μ_j ($j \in \mathbf{N}$) and (ii) ρ_μ is continuous, i.e. vanishing at every singleton, iff μ is continuous. This completes the proof for uniqueness.

Remarks. (1) To prove the existence of a Sobczyk-Hammer decomposition for a charge μ on a field $\mathfrak{A}$ of subsets of a set Ω it also suffices to show that the system $\mathfrak{S} = \{\sum_{i=1}^\infty \mu_i \mid \mu_j$ is an at most two-valued charge on $\mathfrak{A}$ for every $j \in \mathbf{N}$ with $\sum_{i=1}^\infty \mu_i \leqslant \mu\}$ is inductively ordered with respect to the usual partial ordering $\leqslant$ on $\mathrm{ba}(\Omega, \mathfrak{A})$, because in that case Zorn's lemma implies the existence of a maximal element $\bar{\mu}$ in $\mathfrak{S}$. Thus μ can be decomposed into $\mu = \bar{\mu} + \mu'$ where $\mu' = \mu - \bar{\mu} \geqslant 0$. The charge μ' has the property $(*)$.

(2) By the same technique, the following decomposition property can be derived: Let μ be a charge on a field $\mathfrak{A}$ of subsets of a set Ω. Then μ can be decomposed into $\mu = \mu_0 + \mu'$ where μ_0 is a nonatomic charge on $\mathfrak{A}$ and μ' is a charge on $\mathfrak{A}$ which has no nonnull nonatomic minorant, i.e., μ' has the property $(^*_*)$ $\mu' \geqslant \bar{\mu}_0$ implies $\bar{\mu}_0 = 0$ for every nonatomic charge $\bar{\mu}_0$ on $\mathfrak{A}$.

PROOF. Consider the system $\mathfrak{S} = \{\bar{\mu} \mid \bar{\mu}$ is a nonatomic charge on $\mathfrak{A}$ with $\bar{\mu} \leqslant \mu\}$. $\mathfrak{S}$ is inductively ordered with respect to the natural partial ordering $\leqslant$ on $\mathrm{ba}(\Omega, \mathfrak{A})$: Let $\mathfrak{S}_0$ be a subset of $\mathfrak{S}$ which is linearly ordered. Then the charge μ_0 defined by $\mu_0(A) = \sup\{\bar{\mu}(A) \mid \bar{\mu} \in \mathfrak{S}_0\}$ for every $A \in \mathfrak{A}$ is nonatomic and therefore an upper bound for $\mathfrak{S}_0$ in $\mathfrak{S}$. The nonatomicity of μ_0 can be shown by a more canonical indirect argument.

A counterexample shows that in general the preceding decomposition of a charge is not unique:

Define $\Omega = [0, 1]$, $\mathfrak{C} = \{]a, b] \mid]a, b]$ interval with $]a, b] \subset [\frac{1}{4}, \frac{3}{4}[\}$, let $\mathfrak{A}$ denote the field of subsets of Ω which is generated by $\mathfrak{C}$ and let $\mu = \boldsymbol{\lambda} \mid \mathfrak{A}$ be the restriction of the Lebesgue measure $\boldsymbol{\lambda}$ on $\mathfrak{A}$. Then μ is a nonatomic charge on $\mathfrak{A}$. There are two decompositions for μ of the above type: First, choose $\mu_0 = \mu$, μ' vanishing at every $A \in \mathfrak{A}$. Secondly, choose μ' according to $\mu'(B) = \boldsymbol{\lambda}(B \cap A)$ for every $B \in \mathfrak{A}$ where $A = [0, \frac{1}{4}[\cup [\frac{3}{4}, 1]$ and $\mu_0 = \mu - \mu'$. Then the following statements hold: (i) μ' is a charge on $\mathfrak{A}$ with $\mu' \leqslant \mu$, (ii) μ' is two-valued and therefore has the property $(\ast\ast)$ because of the fact that $A \subset B$ or $B \subset [\frac{1}{4}, \frac{3}{4}[$ and thus $\mu'(B) = \frac{1}{2}$ or $\mu'(B) = 0$ for every $B \in \mathfrak{A}$, and (iii) μ_0 is nonatomic.

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REFERENCES

1. G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble) **5** (1955), 131–295.
2. N. Dunford and J. T. Schwartz, *Linear operators*. Part I: *General theory*, Interscience, New York, 1958.
3. R. R. Phelps, *Lectures on Choquet's theorem*, Van Nostrand, Princeton, N. J., 1966.
4. W. Siebe, *Vereibbarkeitsuntersuchungen bei Inhalten und Massen*, Ph. D. Thesis, University of Münster, Münster, 1979.
5. A. Sobczyk and P. C. Hammer, *A decomposition of additive set functions*, Duke Math. J. **11** (1944), 839–846.
6. F. Topsøe, *Topology and measure*, Springer-Verlag, Berlin and New York, 1970.

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