ON THE SOBCZYK-HAMMER DECOMPOSITION
OF ADDITIVE SET FUNCTIONS

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Abstract. It is observed that continuity for charges is equivalent to the absence of two-valued minorants. This characterization forms the basis of a new short proof within a functional-analytic context of a decomposition theorem by A. Sobczyk and P. C. Hammer [5] for charges on a field $\mathcal{A}$ into a continuous part and a part which can be written as a sum of at most two-valued charges on $\mathcal{A}$. A counterexample shows that in general the decomposition of a charge into a nonatomic part and a part which has no nonnull nonatomic minorant is not unique.

1. Definitions. Let $\mathcal{A}$ be a field of subsets of a set $\Omega$. A charge $\mu$ on $\mathcal{A}$ is a real-valued nonnegative finitely additive function defined on $\mathcal{A}$. A measure is a countably additive charge whose domain is a $\sigma$-field of subsets of a set. A set $A \in \mathcal{A}$ will be called an atom for $\mu$ iff $\mu(A) > 0$ and for every set $E \in \mathcal{A}$, $E \subset A$, either $\mu(E) = 0$ or $\mu(E) = \mu(A)$. $\mu$ is nonatomic iff there are no atoms for $\mu$. Finally, a charge $\mu$ on a field $\mathcal{A}$ (in $\Omega$) is said to be continuous iff, given $\varepsilon > 0$, there exists a partition $P = \{B_1, \ldots, B_n\}$ of $\Omega$ into a finite number of pairwise disjoint members of $\mathcal{A}$ such that $\mu(B_i) < \varepsilon$ for every $i$. Let $\mathcal{P}$ denote the family of all such partitions, and $|P(\mu)|$ the maximum of $\mu(B_i)$ over all parts $B_i$ of $P \in \mathcal{P}$.

2. Main result. It is well known that a measure $\lambda$ on a $\sigma$-field $\mathcal{G}$ can be decomposed uniquely into $\lambda = \lambda_0 + \lambda'$ where $\lambda'$ is a nonatomic measure on $\mathcal{G}$ and $\lambda_0$ a completely atomic measure on $\mathcal{G}$, i.e., $\lambda_0$ has the form $\lambda_0 = \sum_{i=1}^{\infty} \lambda_i$ with at most two-valued measures $\lambda_i$ on $\mathcal{G}$ ($i \in \mathbb{N}$).

A. Sobczyk and P. C. Hammer have proved an analogous decomposition theorem for charges on fields of subsets of a set:

Decomposition Theorem (Sobczyk-Hammer [5, Theorem 4.1]). Let $\mu$ be a charge on a field $\mathcal{A}$ (in a set $\Omega$). Then $\mu$ can be decomposed uniquely into $\mu = \mu_0 + \mu'$ with a continuous charge $\mu'$ on $\mathcal{A}$ and a charge $\mu_0$ on $\mathcal{A}$ which has the form $\mu_0 = \sum_{i=1}^{\infty} \mu_i$ with at most two-valued charges $\mu_i$ on $\mathcal{A}$ ($i \in \mathbb{N}$).

The following lemma yields a reformulation of this theorem, for which a short functional-analytic proof will be given. The Krein-Milman representation theorem will serve as a link to the classical Lebesgue decomposition theorem.

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Lemma. Let \( v \) be a charge on a field \( \mathfrak{A} \). \( v \) is continuous iff \( v \) admits no two-valued minorant, i.e., \( v \) has the property (*) \( v \geq \eta \) implies \( \eta = 0 \) for any at most two-valued charge \( \eta \) on \( \mathfrak{A} \).

Proof. A canonical indirect argument shows that a continuous charge on \( \mathfrak{A} \) also has the property (*). Now suppose that \( v \) is not continuous. Then by Lemmas 4.1 and 4.2 of [5] there exists a two-valued charge \( v^* \) on \( \mathfrak{A} \) with \( v \geq v^* \)

Theorem. Let \( \mu \) be a charge on a field \( \mathfrak{A} \) of subsets of a set \( \Omega \). Then \( \mu \) is
decomposable uniquely into \( \mu = \mu_0 + \mu' \) where \( \mu' \) is a charge on \( \mathfrak{A} \) with property (*) and \( \mu_0 \) is a charge on \( \mathfrak{A} \) of the form \( \mu_0 = \sum_{i=1}^{\infty} \mu_i \) with at most two-valued charges \( \mu_i \) on \( \mathfrak{A} \) (\( i \in \mathbb{N} \)).

Proof. Existence. Let \( \mu \) be a probability charge on \( \mathfrak{A} \). Let \( X \) denote the set of all probability charges on \( \mathfrak{A} \) and \( X_\epsilon \) the set of extreme points of \( X \). Then by the theorem of Krein-Milman [3, p. 6] there exists a regular probability measure \( \rho \) on the Borel sets of \( X \) with \( \rho(X_\epsilon) = 1 \) and the property \( \mu(A) = \int_{X_\epsilon} \nu(A) \rho(d\nu) \) for every \( A \in \mathfrak{A} \).

Notice that \( X \) is a convex and by the theorem of Tychonoff compact subset of the locally convex space \( E = \text{ba}(\Omega, \mathfrak{A}) \) of all bounded additive set functions on \( \mathfrak{A} \) under the weak* topology and that the (by [2, V.8.2] nonempty) set \( X_\epsilon \) characterized by the set of the \( \{0,1\} \)-valued charges on \( \mathfrak{A} \) [1, p. 245] is weak* closed. The measure \( \rho \) has a unique decomposition into a convex combination \( \rho = \alpha \rho_1 + (1 - \alpha) \rho_2 \) of a discrete probability measure \( \rho_1 \) and a probability measure \( \rho_2 \) with \( \rho_2(\{v\}) = 0 \) for all \( v \in X \), both on the Borel sets of \( X \) [2, III.4.14]. Thus \( \rho_1 \) can be represented by \( \rho_1 = \sum_{i=1}^{\infty} \alpha_j \delta_\epsilon_i \), where \( \epsilon_i \) denotes the Dirac measure on the Borel sets of \( X \) at \( v \in X_\epsilon \) and where \( \alpha_j \geq 0 \) with \( \sum_{i=1}^{\infty} \alpha_i = 1 \) (\( j \in \mathbb{N} \)). Further, the probability charge \( \nu' \) on \( \mathfrak{A} \) defined by \( \nu'(A) = \int_{X_\epsilon} \nu(A) \rho_2(d\nu) \) for every \( A \in \mathfrak{A} \) has the property (*):

Suppose, there is a two-valued charge \( \tilde{v} \) on \( \mathfrak{A} \) with \( \nu' \geq \tilde{v} \). Define \( \tilde{v} = \nu'/\tilde{\nu}(\Omega) \), \( \Delta = \{ A \in \mathfrak{A} | \tilde{v}(A) = 1 \} \) and \( U_\Delta = \{ v \in X_\epsilon | v(A) = 1 \} \) for every \( A \in \mathfrak{A} \). \( U_\Delta \) is weak* closed, and for every \( A \in \Delta \) the following chain holds:

\[
\nu'(A) = \int_{X_\epsilon} \nu(A) \rho_2(d\nu) = \int_{U_\Delta} \nu(A) \rho_2(d\nu) + \int_{X_\epsilon \setminus U_\Delta} \nu(A) \rho_2(d\nu)
\]

\[
= \rho_2(U_\Delta) \geq \tilde{v}(A) > 0
\]
and therefore $\inf_\Delta \rho_2(U_\Delta) > 0$. Because $\rho_2$ is a bounded regular measure on the Borel sets of a compact space, $\rho_2(\bigcap_\Delta U_\Delta) = \inf_\Delta \rho_2(U_\Delta)$. (Notice that for any $\varepsilon > 0$ there is a weak* compact subset $K$ of $\bigcup_\Delta V_\Delta$ with $V_\Delta = X \setminus U_\Delta$, $A \in \Delta$, such that $\rho_2(K) > \rho_2(\bigcup_\Delta V_\Delta) - \varepsilon$. Thus, there exists a finite family $\{A_1, \ldots, A_n\} \subseteq \Delta$ with $K \subseteq \bigcup_{i=1}^n V_{A_i}$.

Now, define $A_0 = \bigcap_{i=1}^n A_i$. Then, $A_0 \in \Delta$ and $\bigcup_{i=1}^n V_{A_i} \subseteq V_{A_0}$ [the first because of $\bigcap_{i=1}^m U_{A_i} = U_{\bigcap_{i=1}^m B_i}$ for all $m \in \mathbb{N}$, $B_1, \ldots, B_m \in \mathcal{A}$]. Therefore $\rho_2(V_{A_0}) \geq \rho_2(K) > \rho_2(\bigcup_\Delta V_\Delta) - \varepsilon$. Hence $\rho_2(\bigcup_\Delta U_\Delta) = \sup_\Delta \rho_2(U_\Delta')$. Finally, $\bigcap_\Delta U_\Delta = \{\tilde{\mathcal{F}}\}$ holds, which implies $\rho_2(\tilde{\mathcal{F}}) > 0$ in contradiction to the property of $\rho_2$. Then $\mu(A) = \alpha \sum_{i=1}^\infty \alpha_i \nu_i(A) + (1 - \alpha) \nu'(A)$ for every $A \in \mathcal{A}$. Thus the existence of a Sobczyk-Hammer decomposition for $\mu$ is established.

**Uniqueness.** Let $\mu$ be an element of $X$. Then there exists a unique regular probability measure $\rho$ on the Borel sets of $X$ with $\rho(X_\mu) = 1$ which represents $\mu$ (in the sense of [3, p. 2]). Suppose $\tilde{\rho}$ is another measure with the same property. Then $\rho(U_A) = \tilde{\rho}(U_A)$ for every $A \in \mathcal{A}$. Consequently, $\rho$ and $\tilde{\rho}$ coincide on arbitrary unions of sets from $\nabla = \{U_A \mid A \in \mathcal{A}\}$. (Notice that a similar argument as used in the proof for existence shows $\rho(\bigcap_\phi Z_{A_\phi}) = \inf_\phi \rho(Z_{A_\phi})$ with a subfamily $\phi$ of $\mathcal{A}$ which is closed with respect to finite unions, and $Z_{A_\phi} = \{v \in X_\mu \mid \nu(A) = 0\}$, $A \in \phi$. Paying attention to $\rho(X_\mu) = \tilde{\rho}(X_\mu) = 1$, this identity implies the preceding coincidence statement.) Because $X_\mu$ is compact with respect to the relative weak* topology and $\nabla$ is a base for this topology $\rho$ and $\tilde{\rho}$ coincide on the Borel sets of $X_\mu$ and therefore also on the Borel sets of $X$.

It follows that the mapping,

$$X \to \mathcal{M}^+_1(X_\mu),$$

$$\mu \to \rho_\mu,$$

where $\mathcal{M}^+_1(X_\mu)$ is the set of all regular probability measures on the Borel sets of $X$ which are concentrated at $X_\mu$ and where $\rho_\mu$ denotes the regular probability measure which represents $\mu$, is an affine homeomorphism with (i) $\rho_\mu$ is discrete iff $\mu$ has the form $\mu = \sum_{i=1}^\infty \mu_i$ with at most two-valued charges $\mu_j$ ($j \in \mathbb{N}$) and (ii) $\rho_\mu$ is continuous, i.e. vanishing at every singleton, iff $\mu$ is continuous. This completes the proof for uniqueness.

**Remarks.**

(1) To prove the existence of a Sobczyk-Hammer decomposition for a charge $\mu$ on a field $\mathcal{A}$ of subsets of a set $\Omega$ it also suffices to show that the system $\mathcal{S} = \{\sum_{i=1}^\infty \mu_j \mid \mu_j$ is an at most two-valued charge on $\mathcal{A}$ for every $j \in \mathbb{N}$ with $\sum_{i=1}^\infty \mu_i \leq \mu\}$ is inductively ordered with respect to the usual partial ordering $\leq$ on $\text{ba}(\Omega, \mathcal{A})$, because in that case Zorn's lemma implies the existence of a maximal element $\tilde{\mu}$ in $\mathcal{S}$. Thus $\mu$ can be decomposed into $\mu = \tilde{\mu} + \mu'$ where $\mu' = \mu - \tilde{\mu} > 0$.

The charge $\mu'$ has the property $(\star)$.

(2) By the same technique, the following decomposition property can be derived: Let $\mu$ be a charge on a field $\mathcal{A}$ of subsets of a set $\Omega$. Then $\mu$ can be decomposed into $\mu = \mu_0 + \mu'$ where $\mu_0$ is a nonatomic charge on $\mathcal{A}$ and $\mu'$ is a charge on $\mathcal{A}$ which has no nonnull nonatomic minorant, i.e., $\mu'$ has the property $(\star)$ $\mu' > \tilde{\mu}_0$ implies $\tilde{\mu}_0 = 0$ for every nonatomic charge $\tilde{\mu}_0$ on $\mathcal{A}$. 
PROOF. Consider the system \( \mathcal{S} = \{ \bar{\mu} \mid \bar{\mu} \text{ is a nonatomic charge on } \mathfrak{A} \text{ with } \bar{\mu} \subseteq \mu \} \). \( \mathcal{S} \) is inductively ordered with respect to the natural partial ordering \( \leq \) on \( ba(\Omega, \mathfrak{A}) \): Let \( \mathcal{S}_0 \) be a subset of \( \mathcal{S} \) which is linearly ordered. Then the charge \( \mu_0 \) defined by \( \mu_0(A) = \sup(\bar{\mu}(A) \mid \bar{\mu} \in \mathcal{S}_0) \) for every \( A \in \mathfrak{A} \) is nonatomic and therefore an upper bound for \( \mathcal{S}_0 \) in \( \mathcal{S} \). The nonatomicity of \( \mu_0 \) can be shown by a more canonical indirect argument.

A counterexample shows that in general the preceding decomposition of a charge is not unique:

Define \( \Omega = [0, 1] \), \( \mathcal{C} = \{ ]a, b] \mid ]a, b] \text{ interval with } ]a, b] \subseteq [\frac{1}{4}, \frac{3}{4}] \} \), let \( \mathfrak{A} \) denote the field of subsets of \( \Omega \) which is generated by \( \mathcal{C} \) and let \( \mu = \lambda \mid \mathfrak{A} \) be the restriction of the Lebesgue measure \( \lambda \) on \( \mathfrak{A} \). Then \( \mu \) is a nonatomic charge on \( \mathfrak{A} \). There are two decompositions for \( \mu \) of the above type: First, choose \( \mu_0 = \mu \), \( \mu' \) vanishing at every \( A \in \mathfrak{A} \). Secondly, choose \( \mu' \) according to \( \mu'(B) = \lambda(B \cap A) \) for every \( B \in \mathfrak{A} \) where \( A = [0, \frac{1}{4}] \cup [\frac{1}{4}, 1] \) and \( \mu_0 = \mu - \mu' \). Then the following statements hold: (i) \( \mu' \) is a charge on \( \mathfrak{A} \) with \( \mu' \ll \mu \), (ii) \( \mu' \) is two-valued and therefore has the property \((\sharp)\) because of the fact that \( A \subseteq B \text{ or } B \subseteq [\frac{1}{4}, \frac{3}{4}] \) and thus \( \mu'(B) = \frac{1}{2} \) or \( \mu'(B) = 0 \) for every \( B \in \mathfrak{A} \), and (iii) \( \mu_0 \) is nonatomic.

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REFERENCES


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