PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 3, November 1982

UNIT LEMNISCATES CONTAINED IN THE UNIT BALL

MAU-HSIANG SHIH AND HANN-TZONG WANG

ABSTRACT. Let $\{A_1, A_2, \ldots, A_\nu\} \equiv A$ be a set of points in E^n. Let $E(A)$ be the set of points in E^n such that $\prod_{k=1}^{\nu} \overline{pA_k} \leqslant 1$ (where $\overline{pA_k}$ denotes the Euclidean distance between p and A_k), and call this set the unit lemniscate with focal set A. It is shown that if the vertices of a regular tetrahedron lie at the distance $\delta \in (0,1)$ from the origin, then they are the foci of a unit lemniscate contained in the open unit ball of E^3 if and only if the sign of $36 - 64\delta + 2\delta^2 - 64\delta^3 + 36\delta^4 + 27\delta^6$ is positive.

Let $f_n(z) = z^n + \cdots$ and $g_m(z) = z^m + \cdots$ be two complex polynomials, and denote by $E(f)$ and $E(g)$ the set of points $|f_n(z)| \leqslant 1$ and $|g_m(z)| \leqslant 1$, respectively. Erdös observed in [1] that neither of these regions can properly contain the other, and Hall noted in [1] that if the regions coincide then $[f_n(z)]^m = [g_m(z)]^n$.

Erdös and Hwang [2] tried to extend this result to E^n as follows: Let $\{A_1, A_2, \ldots, A_\nu\} \equiv A$ be a set of points in E^n. Let $E(A)$ be the set of points in E^n such that $\prod_{k=1}^{\nu} \overline{pA_k} \leqslant 1$ (where $\overline{pA_k}$ denotes the Euclidean distance between p and A_k), and call this set the unit lemniscate with focal set A. (In E^2 = the complex plane, this reduces to $E(f)$, where A is the set of roots of $f(z)$.) Can one such lemniscate properly contain another?

Erdös and Hwang showed that if A and B have the same number of points, A lies in a plane, and $E(A) \subseteq E(B)$, then $A \equiv B$. However they then produced two sets A and B, each consisting of six distinct points in E^3, such that $E(A) \subsetneq E(B)$. They concluded their paper with a question concerning the possibility of extending their theorem to sets of points A all but one of which are in a plane.

By producing a class of intricate examples in E^3, the second author of the present paper showed in [4] that even that extension is not possible. The examples consist of vertices of a regular 2^n-polygon of radius ε, together with a point δ units directly above its center, with $\varepsilon, \delta \in (0,1)$ and n (depending on ε and δ) sufficiently large. Lemniscates having such sets as their focal sets are shown to lie inside the open unit ball of E^3, which shows the possibility of proper inclusion of unit lemniscates in E^3. Recently, Hwang [3] also constructed an example similar to [4], replacing 2^n by n. Note that all these examples are rather complicated and seem unnatural. Furthermore, it seems worthwhile to find the minimum number of the points of a focal set in E^3 such that its unit lemniscate can be properly contained in the open unit ball of

Received by the editors September 28, 1981 and, in revised form, November 16, 1981, January 19, 1982 and April 21, 1982.

1980 *Mathematics Subject Classification*. Primary 30C10.

Key words and phrases. Complex polynomial, focal set, lemniscate, regular tetrahedron, sphere.

0002-9939/82/0000-0388/$01.75

E^3. The aim of this paper is to solve this problem by exploring certain geometric properties of the unit lemniscate with focal set which consists of vertices of a regular tetrahedron.

We shall establish the following:

THEOREM. *If the vertices of a regular tetrahedron lie at the distance* $\delta \in (0,1)$ *from the origin, then they are the foci of a unit lemniscate contained in the open unit ball of* E^3 *if and only if the sign of* $36 - 64\delta + 2\delta^2 - 64\delta^3 + 36\delta^4 + 27\delta^6$ *is positive.*

PROOF. Let the vertices of a regular tetrahedron with circumradius δ have the cylindrical coordinates

$$A_1 = \left(\frac{\sqrt{8}}{3}, 0, -\frac{\delta}{3}\right), \quad A_2 = \left(\frac{\sqrt{8}}{3}, \frac{2\pi}{3}, -\frac{\delta}{3}\right),$$

$$A_3 = \left(\frac{\sqrt{8}}{3}, \frac{-2\pi}{3}, -\frac{\delta}{3}\right), \quad \text{and} \quad A_4 = (0, 0, \delta).$$

Let S^2 denote the unit sphere of E^3. With the cylindrical coordinate $p = (r, \theta, \pm\sqrt{1-r^2})$ in S^2, let $F(p) = \prod_{k=1}^{4} \overline{pA_k}^2 = d_1^2 d_2^2 d_3^2 d_4^2$, where $d_k = \overline{pA_k}$ $(k = 1, 2, 3, 4)$. We use the Pythagorean formula, and we find that the squares of the four distances are

$$\begin{aligned}
d_1^2 &= 1 + \delta^2 - \frac{4\sqrt{2}\,\delta r\cos\theta}{3} \pm \frac{2\delta\sqrt{1-r^2}}{3}, \\
d_2^2 &= 1 + \delta^2 + \frac{2\sqrt{2}\,\delta r\cos\theta}{3} \pm \frac{2\delta\sqrt{1-r^2}}{3} - \frac{2\sqrt{6}\,\delta r\sin\theta}{3}, \\
d_3^2 &= 1 + \delta^2 + \frac{2\sqrt{2}\,\delta r\cos\theta}{3} \pm \frac{2\delta\sqrt{1-r^2}}{3} + \frac{2\sqrt{6}\,\delta r\sin\theta}{3}, \\
d_4^2 &= 1 + \delta^2 \mp 2\delta\sqrt{1-r^2}.
\end{aligned}$$

In each line, the upper sign is applicable if and only if p lies on the upper hemisphere.

Assume that the standard meridian is located on the plane $\theta = 0$. Then, by the symmetry of the focal set $\{A_1, A_2, A_3, A_4\} \equiv A$, we need only consider the surface bounded by 0° and 60° EAST on S^2. If we write

$$\lambda = 1 + \delta^2 \pm \frac{2\delta\sqrt{1-r^2}}{3} \quad \text{and} \quad \mu = \frac{2\sqrt{2}\,\delta r\cos\theta}{3},$$

then

$$\begin{aligned}
d_1^2 &= \lambda - 2\mu, \\
d_2^2 d_3^2 &= \lambda^2 + 2\lambda\mu + 4\mu^2 - 8\delta^2 r^2/3, \\
d_1^2 d_2^2 d_3^2 &= \lambda^3 - 8\mu^3 + (2\mu - \lambda)8\delta^2 r^2/3.
\end{aligned}$$

The quantity $d_4^2 = 1 + \delta^2 \mp 2\delta\sqrt{1 - r^2}$ is independent of θ. It follows that

$$\begin{aligned}\frac{\partial}{\partial\theta}F(p) &= \frac{\partial}{\partial\theta}\left(d_1^2 d_2^2 d_3^2 d_4^2\right) \\ &= 8d_4^2(2\delta^2 r^2/3 - 3\mu^2)\frac{\partial\mu}{\partial\theta} = \frac{16}{3}d_4^2\delta^2 r^2(1 - 4\cos^2\theta)\frac{\partial\mu}{\partial\theta}.\end{aligned}$$

This is positive if $0 < \theta < \pi/3$.

We have shown that on each horizontal circle on the unit sphere S^2, the restriction of F to the circle has a minimum on each of the three meridians $\theta = 0, 2\pi/3, 4\pi/3$ and a maximum on each of the three meridians $\theta = \pi/3, \pi, 5\pi/3$. In other words, the restriction assumes its minimum on the great semicircles that are the radial projections of the three nonhorizontal edges of the tetrahedron, and it assumes its maximum on the great semicircles that are the radial projections of the bisectors through $(0, 0, 1)$ of the corresponding three faces of the tetrahedron. Considerations of symmetry now make it clear that F assumes its minimum on S^2 at the radial projections of the four vertices of the tetrahedron, and its maximum at the radial projections of the centroids of the four faces.

We have shown that the minimum value $\phi(\delta)$ of F is assumed at the point $(0, 0, 1)$; in other words, that

$$\begin{aligned}\phi(\delta) &= (1 + \delta^2 + 2\delta/3)^3(1 - 2\delta + \delta^2) \\ &= \left\{(1 + \delta^2)^3 + 2\delta(1 + \delta^2)^2 + \frac{4\delta^2}{9}(1 + \delta^2) + \frac{8\delta^3}{27}\right\}(1 - 2\delta + \delta^2) \\ &= \frac{1}{27}(27 + 36\delta^2 - 64\delta^3 + 2\delta^4 - 64\delta^5 + 36\delta^6 + 27\delta^8).\end{aligned}$$

It follows that

$$27\{\phi(\delta) - 1\} = \delta^2(36 - 64\delta + 2\delta^2 - 64\delta^3 + 36\delta^4 + 27\delta^6).$$

This concludes the proof of our theorem. □

The theorem implies the existence of a number δ_0 in $(0, 1)$ such that the unit lemniscate determined by our tetrahedron lies in the open unit ball of E^3 if and only if $0 < \delta < \delta_0$. To see this, we observe that the polynomial

$$K(\delta) = 36 - 64\delta + 2\delta^2 - 64\delta^3 + 36\delta^4 + 27\delta^6$$

has either one or three zeros in the interval $(0, 1)$, since $K(0) > 0$ and $K(1) < 0$. We consider separately the intervals $[0, 1/3]$, $[1/3, 2/3]$, $[2/3, 1]$.

In the first interval, $K(\delta) > 36 - 64/3 - 64/27 > 0$. In the second interval,

$$K'(\delta) < -64 + 4(2/3) - 3 \cdot 64/3^2 + 4 \cdot 36(2/3)^3 + 6 \cdot 27(2/3)^5 < 0.$$

Therefore K decreases in the interval. Inspection shows that K has a zero in the subinterval $(49/100, 1/2)$.

Because $K'''(\delta)$ changes sign only once in $[0, 1]$, the relations

$$K''(0) > 0, \quad K''(1/3) < 0, \quad K''(2/3) > 0$$

imply that $K'(\delta)$ increases in $[2/3, 1]$. Since $K(1) < 0$, we conclude that K has exactly one zero in $(0, 1)$.

Finally, we are deeply indebted to the referee for his valuable criticisms and unusually detailed suggestions which led to the final form and generality in which the result is presented here.

References

1. P. Erdös, *Advanced problems and solutions* 4229, Amer. Math. Monthly **55** (1948), 171.

2. P. Erdös and J. S. Hwang, *On a geometric property of lemniscates*, Aequationes Math. **17** (1978), 344–347.

3. J. S. Hwang, *A problem on a geometric property of lemniscates*, Proc. Amer. Math. Soc. **82** (1981), 390–392.

4. H. T. Wang, *On a problem of Erdös and Hwang*, Technical report, Chung-Yuan University, 1980.

Department of Mathematics, Chung-Yuan University, Chung-Li, Taiwan, Republic of China