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ON THE INTEGRAL MEANS OF DERIVATIVES OF THE ATOMIC FUNCTION

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ABSTRACT. In this note we give upper and lower estimates on integral means of the atomic function and its derivatives over a circle of radius r as r approaches 1. From this we derive some known and new results.

1. Introduction. In 1971, M. R. Cullen [5] conjectured that $\phi' \notin B^{1/2}$ for any singular inner function ϕ. A counterexample was found by H. A. Allen and C. L. Belna [3]; in fact, the atomic functions $S(z) = \exp[(z+1)/(z-1)]$ satisfies $S' \in B^p$ for all $p < 2/3$ and $S' \notin B^{2/3}$. P. R. Ahern and D. N. Clark [2] generalized this and showed that $\phi' \notin B^{2/3}$ provided that ϕ has a singular factor. Further references are [**1** and **2**].

Here we give good estimates of integral means of derivatives of $S(z)$ (our main result), and use these to find analogues of the above results for the spaces D^p and G^p (to be defined below).

2. Definitions. Let f be an analytic function on the unit disc. We shall use the convenient notation,

$$M_p(r, f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta, \qquad p > 0,$$

$$A(r, f) = \iint_{|z|<r} |f'(z)|^2 \, dx\, dy, \qquad p > 0.$$

The classes D^p, $A^{q,p}$ and G^p are defined by $f \in D^p$ $(p > 0)$ if and only if $\int_0^1 A(r, f)^{p/2}\, dr < +\infty$, $f \in A^{q,p}$ $(q > 0, 0 < p < 1)$ if and only if

$$\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^q (1 - r^2)^{1/p-2} r \, dr\, d\theta < \infty,$$

$f \in G^p$ $(p > 0)$ if and only if $\int_0^1 (\int_0^{2\pi} |f'(re^{i\theta})|\, d\theta)^p \, dr < \infty$.

The classes B^p $(0 < p < 1)$ is defined by $B^p = A^{1,p}$. For some properties of D^p spaces see [**6**, **7**, **9** and **10**], for G^p see [**10** and **11**].

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3. The results. The following theorem will be proven in §4.

THEOREM. *Let $p > 0$ and n be a positive natural number. Then there is positive constant K which depends only on p and n such that*

$$\frac{1}{K}\psi(r) \leqslant M_p(r, S^{(n)}) \leqslant K\psi(r), \qquad r \to 1_-, \tag{1}$$

where $\psi(r)$ denotes $(1-r)^{1/2-np}$ if $p > 1/2n$, $|\log(1-r)|$ if $p = 1/2n$ and 1 if $p < 1/2n$.

An immediate corollary is

PROPOSITION 1. *Let $n \in N$. Then*
(i) $S^{(n)} \in D^p$ *if and only if* $p < 4/(4n+1)$,
(ii) $S^{(n+1)} \in A^{q,p}$ *if and only if* $p < 2/(2q(n+1)+1)$,
(iii) $S^{(n)} \in G^p$ *if and only if* $p < 2/(2n+1)$.

If we set $n = 1$ in part (i) and $n = 0$ in (ii) of this proposition we obtain Theorems 5 and 3 in [**8**]. In particular, we have $S' \notin B^{2/3}$ [**4**].

Proposition 1 implies $S' \in D^p$ if and only if $p < 4/5$ and $S' \in G^p$ if and only if $p < 2/3$. The following Propositions 2 and 3 generalize this fact.

PROPOSITION 2. *If ϕ is an inner function with a singular factor then* (i) $\phi' \notin D^{4/5}$ *and* (ii) $\phi' \notin G^{2/3}$.

PROOF. Let us consider (i). We have proved [**9**] $f \in D^p$ if and only if

$$\sum_0^\infty \left(\sum_{k \in I_n} k^{1-2/p} |a_k|^2 \right)^{p/2} < +\infty,$$

where $f(z) = \sum a_k z^k$ and $I_n = \{k\colon 2^n \leqslant k < 2^{n+1}, k \in N\}$. Hence, $D^p \subset A^{2,p/2}$ for $0 < p \leqslant 2$ and, in particular, $D^{4/5} \subset A^{2,2/5}$. Now (i) follows from Ahern's result [**1**] that $\phi' \notin A^{2,2/5}$ if ϕ is an inner function with a singular factor.

Part (ii) follows from the relation $G^p \subset B^p$, $0 < p < 1$ [**10**], and Ahern-Clark's result that $\phi' \notin B^{2/3}$ if ϕ is an inner function with a singular factor.

We need the following definition [**2**]: a compact subset E of $[0, 2\pi]$ is of type β $(0 < \beta \leqslant 1)$ if there is a constant c such that $|E_\varepsilon| \leqslant c\varepsilon^\beta$, where $E_\varepsilon = \{\theta\colon \operatorname{dist}(\theta, E) < \varepsilon\}$. It is clear that E is finite if and only if E is type 1.

PROPOSITION 3. *Suppose σ is a singular measure whose support is a set of type β $(\beta > 0)$. Let ϕ be the corresponding singular inner function. Then*
(i) $\phi' \in D^p$ *for all* $p < 4/(6-\beta)$,
(ii) $\phi' \in G^p$ *for all* $p < 2/(4-\beta)$.

PROOF. Let us first consider (i). Ahern [**1**, Lemmas 4.1 and 5.1] has proved

$$\phi' \in B^p \quad \text{if and only if} \quad \int_0^1 M_2(r, \phi')(1-r)^{1/p-1}\,dr < \infty \qquad \left(\tfrac{1}{2} < p < 1\right). \tag{2}$$

Hence

$$\phi' \in B^p \Rightarrow M_2(r, \phi') = O(1-r)^{-1/p}, \qquad r \to 1_- \ \left(\tfrac{1}{2} < p < 1\right). \tag{3}$$

Combining (2) and (3) with Theorem 4 of [2], we get (i).

For the proof of (ii) it is enough to note $M_1(r,\phi') \leqslant c(1-r)^{q-1}$ for all $q > \beta/2$ (cf. [**2**, Theorem 4]).

4. The proof of the Theorem. The letter "C" in the following should be read "an arbitrary constant, depending only on p and n".

Let $p > 0$ and n a positive natural number. Induction gives

$$S^{(n)}(z) = \frac{P_n(z)}{(z-1)^{2n}} S(z),$$

where P_n is a polynomial and $P_n(1) \neq 0$. Hence, we obtain

$$(4) \quad |S^{(n)}(re^{i\theta})| = \frac{|P_n(re^{i\theta})|}{(1+r^2-2r\cos\theta)^n} \exp\left(-\frac{1-r^2}{1+r^2-2r\cos\theta}\right)$$

$$(0 < r < 1, 0 \leqslant \theta \leqslant 2\pi).$$

From (4) and the inequality $e^{-x} \geqslant 1 - x$ it follows that

$$|S^{(n)}(\mathrm{re}^{i\theta})| \geqslant \frac{r|P_n(\mathrm{re}^{i\theta})|}{2^n(1-r\cos\theta)^{n+1}}(r-\cos\theta) \qquad (r > \cos\theta).$$

Since $P_n(1) \neq 0$, there are positive constants C and r_0, $0 < r_0 < 1$, so that $|P_n(re^{i\theta})| \geqslant C$ for all (r,θ) satisfying $0 \leqslant \theta \leqslant \pi/2$ and $r_0^2 \leqslant \cos\theta \leqslant r^2 < 1$. Hence,

$$|S^{(n)}(re^{i\theta})| \geqslant C\frac{r-\cos\theta}{(1-r\cos\theta)^{n+1}} \qquad \left(0 \leqslant \theta \leqslant \pi/2, r_0^2 \leqslant \cos\theta \leqslant r^2 < 1\right).$$

From this inequality, we obtain

$$M_p(r, S^{(n)}) \geqslant C\int_{r_0^2}^{r^2} \frac{(r-u)^p}{(1-ru)^{np+p}}(1-u)^{-1/2}\,du$$

$$\geqslant C\int_{r_0}^{r^2}(1-ru)^{-np-1/2}\,du \geqslant \frac{1}{K}\psi(r), \qquad r \geqslant r_0.$$

This establishes the left-hand inequality in (1).

For the rest, we note first that, by (4),

$$M_p(r, S^{(n)}) \leqslant C + C\int_0^{\pi/2}(1+r^2-2r\cos\theta)^{-np}\exp\left(-\frac{p(1-r^2)}{1+r^2-2r\cos\theta}\right)d\theta,$$

i.e.

$$(5) \qquad M_p(r, S^{(n)}) \leqslant C + CI_1(r) + CI_2(r),$$

where

$$I_1(r) = \int_0^r g(r,t)^p(1-t^2)^{-1/2}\,dt,$$

$$I_2(r) = \int_r^1 g(r,t)^p(1-t^2)^{-1/2}\,dt$$

and

$$g(r,t) = (1+r^2-2rt)^{-n}\exp\left(-\frac{1-r^2}{1+r^2-2rt}\right).$$

Let $0 \leqslant t \leqslant r$. Then $1 + r^2 - 2rt \geqslant 1 - rt$ and $1 - t^2 > 1 - rt$. Hence,

$$I_1(r) \leqslant \int_0^r (1 - rt)^{-np-1/2}\, dt \leqslant C\psi(r), \qquad r \to 1_-. \tag{6}$$

To estimate $I_2(r)$ we use the equality (derived by direct calculation)

$$\max_{r \leqslant t \leqslant 1} g(r, t) = n^n(1 - r^2)^{-n} e^{-n} \qquad \left(\frac{n-1}{n+1} < r < 1\right). \tag{7}$$

From (7) it follows that

$$I_2(r) \leqslant C(1 - r)^{-np} \int_r^1 (1 - t)^{-1/2}\, dt \leqslant C(1 - r)^{-np+1/2}, \qquad r \to 1_-. \tag{8}$$

Now the right-hand side of (1) follows immediately from (5), (6) and (8).

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