

ON THE INTEGRAL MEANS OF DERIVATIVES OF THE ATOMIC FUNCTION

MIODRAG MATELJEVIĆ AND MIROSLAV PAVLOVIĆ

ABSTRACT. In this note we give upper and lower estimates on integral means of the atomic function and its derivatives over a circle of radius r as r approaches 1. From this we derive some known and new results.

1. Introduction. In 1971, M. R. Cullen [5] conjectured that $\phi' \notin B^{1/2}$ for any singular inner function ϕ . A counterexample was found by H. A. Allen and C. L. Belna [3]; in fact, the atomic functions $S(z) = \exp[(z + 1)/(z - 1)]$ satisfies $S' \in B^p$ for all $p < 2/3$ and $S' \notin B^{2/3}$. P. R. Ahern and D. N. Clark [2] generalized this and showed that $\phi' \notin B^{2/3}$ provided that ϕ has a singular factor. Further references are [1 and 2].

Here we give good estimates of integral means of derivatives of $S(z)$ (our main result), and use these to find analogues of the above results for the spaces D^p and G^p (to be defined below).

2. Definitions. Let f be an analytic function on the unit disc. We shall use the convenient notation,

$$M_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta, \quad p > 0,$$

$$A(r, f) = \iint_{|z| < r} |f'(z)|^2 dx dy, \quad p > 0.$$

The classes D^p , $A^{q,p}$ and G^p are defined by $f \in D^p$ ($p > 0$) if and only if $\int_0^1 A(r, f)^{p/2} dr < +\infty$, $f \in A^{q,p}$ ($q > 0$, $0 < p < 1$) if and only if

$$\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^q (1 - r^2)^{1/p-2} r dr d\theta < \infty,$$

$f \in G^p$ ($p > 0$) if and only if $\int_0^1 (\int_0^{2\pi} |f'(re^{i\theta})| d\theta)^p dr < \infty$.

The classes B^p ($0 < p < 1$) is defined by $B^p = A^{1,p}$. For some properties of D^p spaces see [6, 7, 9 and 10], for G^p see [10 and 11].

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3. The results. The following theorem will be proven in §4.

THEOREM. *Let $p > 0$ and n be a positive natural number. Then there is positive constant K which depends only on p and n such that*

$$(1) \quad \frac{1}{K}\psi(r) \leq M_p(r, S^{(n)}) \leq K\psi(r), \quad r \rightarrow 1_-,$$

where $\psi(r)$ denotes $(1 - r)^{1/2-np}$ if $p > 1/2n$, $|\log(1 - r)|$ if $p = 1/2n$ and 1 if $p < 1/2n$.

An immediate corollary is

PROPOSITION 1. *Let $n \in \mathbb{N}$. Then*

- (i) $S^{(n)} \in D^p$ if and only if $p < 4/(4n + 1)$,
- (ii) $S^{(n+1)} \in A^{q,p}$ if and only if $p < 2/(2q(n + 1) + 1)$,
- (iii) $S^{(n)} \in G^p$ if and only if $p < 2/(2n + 1)$.

If we set $n = 1$ in part (i) and $n = 0$ in (ii) of this proposition we obtain Theorems 5 and 3 in [8]. In particular, we have $S' \notin B^{2/3}$ [4].

Proposition 1 implies $S' \in D^p$ if and only if $p < 4/5$ and $S' \in G^p$ if and only if $p < 2/3$. The following Propositions 2 and 3 generalize this fact.

PROPOSITION 2. *If ϕ is an inner function with a singular factor then (i) $\phi' \notin D^{4/5}$ and (ii) $\phi' \notin G^{2/3}$.*

PROOF. Let us consider (i). We have proved [9] $f \in D^p$ if and only if

$$\sum_0^\infty \left(\sum_{k \in I_n} k^{1-2/p} |a_k|^2 \right)^{p/2} < +\infty,$$

where $f(z) = \sum a_k z^k$ and $I_n = \{k: 2^n \leq k < 2^{n+1}, k \in \mathbb{N}\}$. Hence, $D^p \subset A^{2,p/2}$ for $0 < p \leq 2$ and, in particular, $D^{4/5} \subset A^{2,2/5}$. Now (i) follows from Ahern’s result [1] that $\phi' \notin A^{2,2/5}$ if ϕ is an inner function with a singular factor.

Part (ii) follows from the relation $G^p \subset B^p$, $0 < p < 1$ [10], and Ahern-Clark’s result that $\phi' \notin B^{2/3}$ if ϕ is an inner function with a singular factor.

We need the following definition [2]: a compact subset E of $[0, 2\pi]$ is of type β ($0 < \beta \leq 1$) if there is a constant c such that $|E_\epsilon| \leq c\epsilon^\beta$, where $E_\epsilon = \{\theta: \text{dist}(\theta, E) < \epsilon\}$. It is clear that E is finite if and only if E is type 1.

PROPOSITION 3. *Suppose σ is a singular measure whose support is a set of type β ($\beta > 0$). Let ϕ be the corresponding singular inner function. Then*

- (i) $\phi' \in D^p$ for all $p < 4/(6 - \beta)$,
- (ii) $\phi' \in G^p$ for all $p < 2/(4 - \beta)$.

PROOF. Let us first consider (i). Ahern [1, Lemmas 4.1 and 5.1] has proved

$$(2) \quad \phi' \in B^p \quad \text{if and only if} \quad \int_0^1 M_2(r, \phi')(1 - r)^{1/p-1} dr < \infty \quad \left(\frac{1}{2} < p < 1\right).$$

Hence

$$(3) \quad \phi' \in B^p \Rightarrow M_2(r, \phi') = O(1 - r)^{-1/p}, \quad r \rightarrow 1_- \quad \left(\frac{1}{2} < p < 1\right).$$

Combining (2) and (3) with Theorem 4 of [2], we get (i).

For the proof of (ii) it is enough to note $M_1(r, \phi') \leq c(1 - r)^{q-1}$ for all $q > \beta/2$ (cf. [2, Theorem 4]).

4. The proof of the Theorem. The letter “C” in the following should be read “an arbitrary constant, depending only on p and n ”.

Let $p > 0$ and n a positive natural number. Induction gives

$$S^{(n)}(z) = \frac{P_n(z)}{(z - 1)^{2n}} S(z),$$

where P_n is a polynomial and $P_n(1) \neq 0$. Hence, we obtain

$$(4) \quad |S^{(n)}(re^{i\theta})| = \frac{|P_n(re^{i\theta})|}{(1 + r^2 - 2r \cos \theta)^n} \exp\left(-\frac{1 - r^2}{1 + r^2 - 2r \cos \theta}\right) \quad (0 < r < 1, 0 \leq \theta \leq 2\pi).$$

From (4) and the inequality $e^{-x} \geq 1 - x$ it follows that

$$|S^{(n)}(re^{i\theta})| \geq \frac{r |P_n(re^{i\theta})|}{2^n(1 - r \cos \theta)^{n+1}} (r - \cos \theta) \quad (r > \cos \theta).$$

Since $P_n(1) \neq 0$, there are positive constants C and r_0 , $0 < r_0 < 1$, so that $|P_n(re^{i\theta})| \geq C$ for all (r, θ) satisfying $0 \leq \theta \leq \pi/2$ and $r_0^2 \leq \cos \theta \leq r^2 < 1$. Hence,

$$|S^{(n)}(re^{i\theta})| \geq C \frac{r - \cos \theta}{(1 - r \cos \theta)^{n+1}} \quad (0 \leq \theta \leq \pi/2, r_0^2 \leq \cos \theta \leq r^2 < 1).$$

From this inequality, we obtain

$$\begin{aligned} M_p(r, S^{(n)}) &\geq C \int_{r_0^2}^{r^2} \frac{(r - u)^p}{(1 - ru)^{np+p}} (1 - u)^{-1/2} du \\ &\geq C \int_{r_0}^{r^2} (1 - ru)^{-np-1/2} du \geq \frac{1}{K} \psi(r), \quad r \geq r_0. \end{aligned}$$

This establishes the left-hand inequality in (1).

For the rest, we note first that, by (4),

$$M_p(r, S^{(n)}) \leq C + C \int_0^{\pi/2} (1 + r^2 - 2r \cos \theta)^{-np} \exp\left(-\frac{p(1 - r^2)}{1 + r^2 - 2r \cos \theta}\right) d\theta,$$

i.e.

$$(5) \quad M_p(r, S^{(n)}) \leq C + CI_1(r) + CI_2(r),$$

where

$$\begin{aligned} I_1(r) &= \int_0^r g(r, t)^p (1 - t^2)^{-1/2} dt, \\ I_2(r) &= \int_r^1 g(r, t)^p (1 - t^2)^{-1/2} dt \end{aligned}$$

and

$$g(r, t) = (1 + r^2 - 2rt)^{-n} \exp\left(-\frac{1 - r^2}{1 + r^2 - 2rt}\right).$$

Let $0 \leq t \leq r$. Then $1 + r^2 - 2rt \geq 1 - rt$ and $1 - t^2 > 1 - rt$. Hence,

$$(6) \quad I_1(r) \leq \int_0^r (1 - rt)^{-np-1/2} dt \leq C\psi(r), \quad r \rightarrow 1_-.$$

To estimate $I_2(r)$ we use the equality (derived by direct calculation)

$$(7) \quad \max_{r \leq t \leq 1} g(r, t) = n^n (1 - r^2)^{-n} e^{-n} \quad \left(\frac{n-1}{n+1} < r < 1 \right).$$

From (7) it follows that

$$(8) \quad I_2(r) \leq C(1 - r)^{-np} \int_r^1 (1 - t)^{-1/2} dt \leq C(1 - r)^{-np+1/2}, \quad r \rightarrow 1_-.$$

Now the right-hand side of (1) follows immediately from (5), (6) and (8).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BELGRADE, STUDENTSKI TRG 16, 11000 BELGRADE, YUGOSLAVIA

Current address: Department of Mathematics, University of Wisconsin-Madison, Madison, Wisconsin 53706