ON A GAP TAUBERIAN THEOREM
OF LORENTZ AND ZELLER

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ABSTRACT. G. G. Lorentz and K. L. Zeller have stated an O-Tauberian theorem which places a restriction on the rate of absolute convergence of the row sums of a regular summability method. In this note, we prove a theorem that has as a corollary an extension of the above result in which this restriction is deleted.

By a regular summability method $A = (a_{pq})$ we mean one that sums every convergent sequence $x$ to $\lim_n x_n$. Such methods are characterized by the familiar Silverman-Toeplitz conditions

\begin{align*}
(1) \quad &\lim_{p} a_{pq} = 0 \quad \text{for } q = 1, 2, 3, \ldots, \\
&\lim_{p} \sum_{q} a_{pq} = 1, \quad \text{and} \\
&\sup_{p} \sum_{q} |a_{pq}| \quad \text{is finite.}
\end{align*}

We will consider arbitrary series with terms $u_n$ and partial sums $s_n$. Let $1 = q(0) < q(1) < q(2) < \cdots$ be positive integers and let $0 < G_n$ with $\lim_n G_n = +\infty$. Assume

\begin{align*}
(2) \quad &u_n = 0 \quad \text{if } n \neq q(j), \quad j = 0, 1, 2, 3, \ldots, \quad \text{and} \\
&s_n = O(G_n).
\end{align*}

THEOREM 1 (LORENTZ AND ZELLER [4, p. 402]). Let $A$ be a regular summability method such that

\begin{align*}
(3) \quad &\sum_{q} |a_{pq}| G_q < +\infty \quad \text{for } p = 1, 2, 3, \ldots.
\end{align*}

Then there exist indices $q(1), q(2), \ldots$ for which (2) is a Tauberian condition for $A$. In addition, one can assume the $q(j)$ belong to a given sequence of indices $n(1), n(2), \ldots$.

In our Theorem 2 below, we not only claim condition (c) of (1) is not necessary in Theorem 1, but also that restriction (3) may be omitted, thus assuring independence between $A$ and the sequence $(G_n)$.

THEOREM 2. Let $A$ be a matrix summability method satisfying conditions (a) and (b) of (1). Then there exist indices $q(1), q(2), \ldots$ for which (2) is a Tauberian condition for $A$. In addition, one can assume the $q(j)$ to belong to a given sequence of indices $n(1), n(2), \ldots$.
By a dilution of a series we will mean the insertion of zeros between the terms of the series. If \( s \) is the sequence of partial sums of \( \sum u_n \) and \( \sum v_n \) is a dilution of \( \sum u_n \), then we call the sequence \( t \) of partial sums of \( \sum v_n \) a stretching of \( s \). Dilutions and stretchings are sometimes called gap series and gap sequences respectively. In addition to [4], Tauberian theorems for gap sequences (stretchings) may also be found in [1, 2, and 3].

For each stretching \( t \) of \( s \) there exists a regular matrix \( K \) with all entries either 0 or 1 such that \( Ks = t \). We denote the set of all such stretching transformations as \( \Lambda \) and for \( K \in \Lambda \) let \( q_k(0) = 1 \) and \( q_K(j) = 1 + \max \{i : k_{ij} = 1\} \) for \( j = 1, 2, 3, \ldots \). Theorem 2 may now be rewritten using this notation.

**THEOREM 2A.** Let \( A \) be a matrix summability method satisfying conditions (a) and (b) of (1) and \( n(1), n(2), \ldots \) be an increasing sequence of positive integers. Then there exists \( K \in \Lambda \) with \( q_K(1), q_K(2), \ldots \) a subsequence of \( n(1), n(2), \ldots \) for which condition (b) of (2) and \( A(Ks) \in c \) implies \( s \in c \).

Let \( \epsilon(1), \epsilon(2), \ldots \) be a positive term null sequence. Following Dawson [2], we say the sequence \( x \) contains an \( \epsilon \)-copy of the sequence \( s \) if there exists a subsequence \( y \) of \( x \) such that \( |y_i - s_i| < \epsilon_i \) for \( i = 1, 2, \ldots \). Theorem 2A is a direct consequence of the following result.

**THEOREM 3.** Let \( A \) be a matrix summability method satisfying conditions (a) and (b) of (1) and \( n(1), n(2), \ldots \) be an increasing sequence of positive integers. If \( |s_n| \leq MG_n \) for each \( n \), then there exists \( K \in \Lambda \) with \( q_K(1), q_K(2), \ldots \) a subsequence of \( n(1), n(2), \ldots \) such that \( A(Ks) \) exists and contains an \( \epsilon \)-copy of \( s \).

**PROOF.** Suppose \( 1 = q(0) < q(1) < \cdots < q(i - 1) = n(r) \) and \( 1 \leq p(1) < \cdots < p(i - 1) \) have been determined. We choose \( p(i) > p(i - 1) \) and \( q(i) \in \{n(r + 1), n(r + 2), \ldots\} \) such that

\[
(i) \quad \sum_{j=1}^{q(i)-1} MG(j) \sum_{q=q(j-1)}^{q(i)-1} a_{pq} < \epsilon_i / 2 \quad \text{whenever} \quad p \geq p(i),
\]

\[
(ii) \quad MG(i) \sum_{q=q(i-1)}^{q(i)-1} a_{p(i),q} - 1 < \epsilon_i / 4, \quad \text{and}
\]

\[
(iii) \quad MG(i+1) \sup_{j,p \leq p(i)} \left| \sum_{q=q(i)}^{q(j)} a_{pq} \right| < \max_{j \leq i} \epsilon_j / 2^{i+2}.
\]

The sequence \( q(0), q(1), q(2), \ldots \) thus uniquely determines a \( K \in \Lambda \) such that \( A(Ks) \) exists, and if \( Ks = t \), then

\[
\sum_{q=1}^{\infty} a_{p(q),q} t_q - s_i \leq \epsilon_i
\]

for \( i = 1, 2, 3, \ldots \).

**REFERENCES**


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