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ON A GAP TAUBERIAN THEOREM OF LORENTZ AND ZELLER

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ABSTRACT. G. G. Lorentz and K. L. Zeller have stated an O-Tauberian theorem which places a restriction on the rate of absolute convergence of the row sums of a regular summability method. In this note, we prove a theorem that has as a corollary an extension of the above result in which this restriction is deleted.

By a regular summability method $A = (a_{pq})$ we mean one that sums every convergent sequence x to $\lim_n x_n$. Such methods are characterized by the familiar Silverman-Toeplitz conditions

$$(1) \qquad \begin{aligned} &\text{(a)} \quad \lim_p a_{pq} = 0 \quad \text{for } q = 1,2,3,\ldots, \\ &\text{(b)} \quad \lim_p \sum_q a_{pq} = 1, \quad \text{and} \\ &\text{(c)} \quad \sup_p \sum_q |a_{pq}| \quad \text{is finite.} \end{aligned}$$

We will consider arbitrary series with terms u_n and partial sums s_n. Let $1 = q(0) < q(1) < q(2) < \cdots$ be positive integers and let $0 < G_n$ with $\lim_n G_n = +\infty$. Assume

$$(2) \qquad \begin{aligned} &\text{(a)} \quad u_n = 0 \quad \text{if } n \neq q(j),\ j = 0,1,2,3,\ldots, \text{ and} \\ &\text{(b)} \quad s_n = O(G_n). \end{aligned}$$

THEOREM 1 (LORENTZ AND ZELLER [4, p. 402]). *Let A be a regular summability method such that*

$$(3) \qquad \sum_q |a_{pq}| G_q < +\infty \quad \textit{for } p = 1,2,3,\ldots.$$

Then there exist indices $q(1)$, $q(2), \ldots$ for which (2) *is a Tauberian condition for A. In addition, one can assume the $q(j)$ belong to a given sequence of indices $n(1)$, $n(2), \ldots$.*

In our Theorem 2 below, we not only claim condition (c) of (1) is not necessary in Theorem 1, but also that restriction (3) may be omitted, thus assuring independence between A and the sequence (G_n).

THEOREM 2. *Let A be a matrix summability method satisfying conditions* (a) *and* (b) *of* (1). *Then there exist indices $q(1)$, $q(2), \ldots$ for which* (2) *is a Tauberian condition for A. In addition, one can assume the $q(j)$ to belong to a given sequence of indices $n(1)$, $n(2), \ldots$.*

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By a dilution of a series we will mean the insertion of zeros between the terms of the series. If s is the sequence of partial sums of $\sum u_n$ and $\sum v_n$ is a dilution of $\sum u_n$, then we call the sequence t of partial sums of $\sum v_n$ a stretching of s. Dilutions and stretchings are sometimes called gap series and gap sequences respectively. In addition to [**4**], Tauberian theorems for gap sequences (stretchings) may also be found in [**1**, **2**, and **3**].

For each stretching t of s there exists a regular matrix K with all entries either 0 or 1 such that $Ks = t$. We denote the set of all such stretching transformations as Λ and for $K \in \Lambda$ let $q_k(0) = 1$ and $q_K(j) = 1 + \max\{i : k_{ij} = 1\}$ for $j = 1, 2, 3, \ldots$. Theorem 2 may now be rewritten using this notation.

THEOREM 2A. *Let A be a matrix summability method satisfying conditions* (a) *and* (b) *of* (1) *and $n(1)$, $n(2), \ldots$ be an increasing sequence of positive integers. Then there exists $K \in \Lambda$ with $q_K(1)$, $q_K(2), \ldots$ a subsequence of $n(1)$, $n(2), \ldots$ for which condition* (b) *of* (2) *and $A(Ks) \in c$ implies $s \in c$.*

Let $\epsilon(1)$, $\epsilon(2), \ldots$ be a positive term null sequence. Following Dawson [**2**], we say the sequence x contains an ϵ-copy of the sequence s if there exists a subsequence y of x such that $|y_i - s_i| < \epsilon_i$ for $i = 1, 2, \ldots$. Theorem 2A is a direct consequence of the following result.

THEOREM 3. *Let A be a matrix summability method satisfying conditions* (a) *and* (b) *of* (1) *and $n(1)$, $n(2), \ldots$ be an increasing sequence of positive integers. If $|s_n| \leq MG_n$ for each n, then there exists $K \in \Lambda$ with $q_K(1)$, $q_K(2), \ldots$ a subsequence of $n(1)$, $n(2), \ldots$ such that $A(Ks)$ exists and contains an ϵ-copy of s.*

PROOF. Suppose $1 = q(0) < q(1) < \cdots < q(i-1) = n(r)$ and $1 \leq p(1) < \cdots < p(i-1)$ have been determined. We choose $p(i) > p(i-1)$ and $q(i) \in \{n(r+1), \quad n(r+2), \ldots\}$ such that

(i) $\sum_{j=1}^{i-1} MG(j) \left| \sum_{q=q(j-1)}^{q(j)-1} a_{pq} \right| < \epsilon_i/2$ whenever $p \geq p(i)$,

(ii) $MG(i) \left| \sum_{q=q(i-1)}^{q(i)-1} a_{p(i),q} - 1 \right| < \epsilon_i/4$, and

(iii) $MG(i+1) \sup_{j,p \leq p(i)} \left| \sum_{q=q(i)}^{j} a_{pq} \right| < \max_{j \leq i} \epsilon_j / 2^{i+2}$.

The sequence $q(0)$, $q(1)$, $q(2), \ldots$ thus uniquely determines a $K \in \Lambda$ such that $A(Ks)$ exists, and if $Ks = t$, then

$$\left| \sum_{q=1}^{\infty} a_{p(i),q} t_q - s_i \right| < \epsilon_i$$

for $i = 1, 2, 3, \ldots$.

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