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APPROXIMATE TOPOLOGY ON Rep(A)

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ABSTRACT. Let A be a C^*-algebra and Rep(A) the set of all nondegenerate representations of A. We define a new topology on Rep(A) and study the relations with the point weak topology of Rep(A).

1. Introduction Throughout the paper, let A be a nonzero C^*-algebra, and let $P(A)$, $\hat{A}$ and Prim(A) denote, respectively, the pure states, the unitary equivalence classes of all nonzero irreducible representations and the set of all primitive ideals of A. As usual, we consider Prim(A) as a topological space with the Jacobson topology [**3**, p. 70]. The spectrum $\hat{A}$ is topologized with the inverse image of the Jacobson topology under the canonical surjection $\alpha_1 \colon \hat{A} \to \mathrm{Prim}(A)$, sending the unitary equivalence class $[\pi]$ of a unitary representation π to the kernel $\ker(\pi)$ of π. Thus α_1 is a continuous open surjection. Prim(A) is a T_0-space [**3**, p. 70] and not Hausdorff in general. On the other hand, the spectrum $\hat{A}$ is not even a T_0-space in the general situation [**3**, p. 71]. We regard $P(A)$ as the topological space relativised from the weak* topology $\sigma(A^*, A)$ on the norm dual space A^* of A. For any $f \in P(A)$, let π_f be the irreducible representation of A associated with f, under the Gelfand-Naimark-Segal construction. The mapping $\alpha_2 \colon f \to [\pi_f]$ is an open and continuous surjection [**3**, p. 79], but it is many-to-one, [**3**, p. 54].

Let Rep(A) denote the set of all nondegenerate representations of A on nonzero Hilbert spaces. We will consider the weak topology $\mathcal{T}_w$ on Rep(A), which is essentially the same as the topology on Rep($A : H$) of M. Takesaki [**10**, p. 376] or the strong topology of L. T. Gardner [**5**, p. 445]. Let Irr(A) be the set of all nonzero irreducible representations of A. Let $\sim$ be the approximate equivalence in Rep(A) [**1**, **6**]. In the main theorem (Theorem), the set of equivalence classes in Rep(A) under $\sim$, equipped with the quotient topology $\tilde{\mathcal{T}}_w$, will be shown to be homeomorphic with Prim(A). But our principal concern of this paper is to introduce a new topology $\mathcal{T}$ on Rep(A), called the approximate topology, which will induce a Hausdorff topology on Prim(A) stronger than the Jacobson topology. This result is also contained in the Theorem.

2. The approximate topology For brevity, we write X for Rep(A). If $\pi \in X$, then we denote by H_π the representation space of π. For any two $\pi, \rho \in X$, we say that they are *approximately equivalent*, denoted by $\pi \sim \rho$, if there is a net $\{U_i\}$ of unitary operators $U_i \colon H_\rho \to H_\pi$ such that

$$||U_i^* \pi(a) U_i - \rho(a)|| \to 0, \quad \text{for every } a \in A$$

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[**1**, **6**, **7**]. Truly it gives an equivalence relation in X.

Let $\mathcal{F}$ denote the family of all nonempty finite subsets of A. We define

$$(1)\quad V_{F,\delta} = \{(\pi,\rho) \in X \times X : \|U^*_{\pi\rho}\pi(a)U_{\pi\rho} - \rho(a)\| < \delta, \text{ for all } a \in F, \text{ for some unitary } U_{\pi\rho}\colon H_\rho \to H_\pi\}.$$

Here $U_{\pi\rho}$ depends upon F, δ and (π,ρ), but not on the choice of a from F, once F is determined. Let

$$(2)\qquad \mathcal{B}_F = \{V_{F,\delta} : \delta > 0\}.$$

Then, clearly each $V_{F,\delta}$ contains the diagonal of $X \times X$ and $V^{-1}_{F,\delta} \in \mathcal{B}_F$, whenever $V_{F,\delta} \in \mathcal{B}_F$. Also we have

$$(3)\qquad V_{F,\delta/2} \circ V_{F,\delta/2} \in V_{F,\delta}.$$

In fact, if $\|U^*_{\pi\rho}\pi(a)U_{\pi\rho} - \rho(a)\| < \delta/2$ and $\|U^*_{\rho\sigma}\rho(a)U_{\rho\sigma} - \sigma(a)\| < \sigma/2$, for all $a \in F$, then,

$$\begin{aligned}\|(U_{\pi\rho}U_{\rho\sigma})^*\pi(a)U_{\pi\rho}U_{\rho\sigma} - U^*_{\rho\sigma}\rho(a)U_{\rho\sigma}\| &+ \|U^*_{\rho\sigma}\rho(a)U_{\rho\sigma} - \sigma(a)\| \\ &\le \|U^*_{\pi\rho}\pi(a)U_{\pi\rho} - \rho(a)\| + \|U^*_{\rho\sigma}\rho(a)U_{\rho\sigma} - \sigma(a)\| \\ &< \delta/2 + \delta/2 = \delta, \quad \text{for all } a \in F.\end{aligned}$$

Thus, the fact that $(\pi,\rho) \in V_{F,\delta/2}$ and $(\rho,\sigma) \in V_{F,\delta/2}$ imply that $(\pi,\sigma) \in V_{F,\delta}$.

Clearly,

$$(4)\qquad V_{F,\delta} \subset V_{F,\delta} \cap V_{F,\epsilon}, \quad \text{where } \alpha = \min(\delta,\epsilon).$$

It follows that $\mathcal{B}_F$ forms a fundamental system of entourages for a unique uniformity $\mathcal{U}_F$ on X [**2**, p. 170]. Let $\mathcal{U}$ denote the least upper bound of $\{\mathcal{U}_F : F \in \mathcal{F}\}$ in the ordered set of all uniformities on X [**2**, p. 178]. We put

$$(5)\qquad R = \bigcap\{V : V \in \mathcal{U}\}.$$

Then, it is not hard to show that

$$(6)\qquad (\pi,\rho) \in R \quad \text{if and only if } \pi \sim \rho$$

[**6**, Lemma 2.4].

DEFINITION 1. The topology $\mathcal{T}$ on X associated with the uniformity $\mathcal{U}$ is called the *approximate topology* on X.

Now let $(\tilde{X}, \tilde{\mathcal{T}})$ be the Hausdorff uniform space associated with the unique separated uniformity $\tilde{\mathcal{U}}$ on the quotient set $\tilde{X} = X/R$ [**8**, p. 28, Theorem 3.16], so that the quotient mapping $(X,\mathcal{U})/(\tilde{X},\tilde{\mathcal{U}})$ is a uniformly continuous, open and closed surjection.

DEFINITION 2. Let $\mathrm{Irr}(A)$ denote the set of all nonzero irreducible representations of A and $\tilde{A}$ denote the uniform space $\mathrm{Irr}(A)R$ relativised from $(\tilde{X}, \tilde{\mathcal{T}})$ [**2**, p. 178]. We call $\tilde{A}$ the *approximate dual* of A.

For $\pi \in X$, $K = \bigoplus\{\mathcal{H}_\pi : \pi \in X\}$ and $\jmath(\pi)\colon H_\pi \hookrightarrow K$, let us define

$$\tilde{\pi}(x) = \jmath(\pi)\pi(x)\jmath(\pi)^*, \quad \text{for each } x \in A.$$

Let $\mathcal{T}_w$ denote the inverse image of the point weak topology of $\mathrm{Rep}(A:K)$, where $\mathrm{Rep}(A:K)$ is the set of all nonzero representations of A on K, [**10**, p. 376; **3**, p. 80; **5**, p. 445] under the injection $\pi \to \tilde{\pi}\colon X \to \mathrm{Rep}(A:K)$. We call $\mathcal{T}_w$ the *weak topology* on X.

We consider the next diagram of mappings of which definitions are omitted, since they are too obvious. (Irr(A), $\mathcal{T}_w$) is the relativised topological space of $(X, \mathcal{T}_w)$, and $(\tilde{A}, \tilde{\mathcal{T}}_w)$ is the quotient space of (Irr(A), $\mathcal{T}_w$) under the approximate equivalence, while β_1 is the quotient mapping. Thus β_1 as well as α_1 and α_2 is an identification map [**4**, p. 121, 1.3 Definition].

$$\begin{tikzcd}
P(A) \arrow[rr, "\beta_2"] \arrow[dr, "\alpha_2"] & & (\mathrm{Irr}(A), \mathcal{T}) \arrow[dl, "\gamma_2"'] \arrow[d, "\beta_1"] & & (\mathrm{Irr}(A), \mathcal{T}) \arrow[ll] \arrow[dr, "\beta_1"] & \\
& \hat{A} \arrow[r] \arrow[dr, "\alpha_1"'] & (\tilde{A}, \tilde{\mathcal{T}}_w) \arrow[d, "\gamma"] & & & (\tilde{A}, \tilde{\mathcal{T}}) \arrow[lll, "\gamma_1"'] \arrow[dlll, "\gamma"'] \\
& & \mathrm{Prim}(A) & & &
\end{tikzcd}$$

It is known that two irreducible representations with the same kernel are approximately equivalent [**7**, p. 337; **6**, p. 10, *Approximate versus unitary equivalent*]. To see that γ is continuous for $\tilde{\mathcal{T}}_w$, it suffices to show that $\gamma \circ \beta_1$ is continuous for $\mathcal{T}_w$. Since $\gamma \circ \beta_1 = \alpha_1 \circ \gamma_2$, it is then enough to show that γ_2 is continuous. But γ_2 is well known to be a continuous open surjection [**5**, p. 445]. It follows that γ is in fact a continuous open bijection in $\tilde{\mathcal{T}}_w$, also that γ is a continuous bijection for $\tilde{\mathcal{T}}$. We summarize these discussions as follows.

THEOREM. *γ is a homeomorphism with respect to $\tilde{\mathcal{T}}_w$ and it is a continuous bijection with respect to $\tilde{\mathcal{T}}$.*

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