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ON THE RELATION BETWEEN LEFT THICKNESS AND TOPOLOGICAL LEFT THICKNESS IN SEMIGROUPS

JAMES C. S. WONG[1]

ABSTRACT. In this paper, we establish an interesting relation between left thickness and topological left thickness in semigroups by showing that a Borel subset T of a locally compact semigroup S is topological left thick in S iff a certain subset M_T associated with T is left thick in a semigroup S_1 containing S, or equivalent, iff M_T contains a left ideal of S_1. Our results contain a topological analogue of a result of H. Junghenn in [*Amenability of function spaces on thick subsemigroups*, Proc. Amer. Math. Soc. **75** (1979), 37–41]. However, even in the case of discrete semigroups, our results are more general and in a way more natural than those of Junghenn's. Furthermore, the fact that M_T is left thick iff it contains a left ideal in S_1 is quite surprising, since in general, a left thick subset need not contain a left ideal although the converse is always true.

1. Introduction. For terminologies in analysis on semigroups, we shall follow Junghenn [**4**] and Wong [**13**]. Also, this paper is a sequel to Wong [**13**] and we shall freely use the notations there.

Let S be a locally compact (separately continuous) semigroup with convolution measure algebra $M(S)$ and let $M_0(S)$, as usual, be the convolution semigroup of probability measures on S. Also let $X \subset M(S)^*$ be a linear subspace of $M(S)^*$ containing the constant functional 1. Throughout this paper, we assume that X is topological left invariant: $\mu \odot X \subset X$ for any $\mu \in M(S)$ where $\mu \odot F(\nu) = F(\mu * \nu)$, $F \in X$, $\nu \in M(S)$. X is called topological left introverted if $M_L(X) \subset X$ for all $M \in M(S)^{**}$ where

$$M_L(F)(\mu) = M(\mu \odot F), \qquad F \in X,\ \mu \in M(S).$$

If T is a Borel subset of S, we define the characteristic functional χ_T of T by $\chi_T(\mu) = \mu(T)$, $\mu \in M(S)$ (so that $\chi_S = 1$) while (unlike in Junghenn [**4**]) the characteristic function of T is denoted by ξ_T (as in Wong [**13**]).

We say that T is topological left X-thick in S iff for each triple (ϵ, μ, F) where $\epsilon > 0$, $\mu \in M_0(S)$ with compact support and $F \in X$, $\chi_T \le F \le 1$, there is some $\nu \in M_0(S)$ with compact support such that $F(\mu * \nu) > 1 - \epsilon$.

Notice that when $X = M(S)^*$, this is exactly the definition of left lumpiness introduced by Day [**1**] but is weaker than the concept of topological left thickness

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as defined in Wong [**11**, condition (TLT), §2, p. 572]. Of course, it is now clear that the above definition is the "proper" one for topological left thickness while condition (TLT) in Wong [**11**] should really be called "uniform" topological left thickness. (See Wong [**10**, Theorem 4.1] and Wong [**11**, Addendum, p. 585].)

For each $\mu \in M_0(S)$, define $Q(\mu) \in X^*$ by $Q(\mu)(F) = F(\mu)$, $F \in X$. It is well known that each $Q(\mu)$ is a mean on X and that the set of all such means is weak* dense in the set $M(X)$ of all means on X.

2. Some preliminary results. The next theorem is a topological analogue and extension of results in Junghenn [**4**] and Wong [**13**]. Note that we do not require X to have any algebra structure (nor to be norm closed) unlike in [**4**]. In fact, $M(S)^*$ is not an algebra with respect to pointwise product.

THEOREM 2.1. *The following statements on T are equivalent:*

(a) *T is topological left X-thick.*

(b) *There is some mean M on X such that $M(\mu \odot F) = 1$ for all $\mu \in M_0(S)$ and $F \in X$ with $\chi_T \leq F \leq 1$.*

If in addition, X is topological left introverted, then both are equivalent to

(c) *There is a mean M on X such that $N \odot M(F) = 1$ for all mean N on X and all $F \in X$ with $\chi_T \leq F \leq 1$.*

Here $N \odot M$ is the Arens product in X^*.

PROOF. (a) implies (b) follows from a topological version of the proof in Junghenn [**4**, Theorem 1, (b) implies (c), p. 39]. We omit the details. Conversely, assume (b). If T is not topological left X-thick, then there is a triple (ϵ_0, μ_0, F_0) such that $F_0(\mu_0 * \nu) \leq 1 - \epsilon_0$ for any $\nu \in M_0(S)$ with compact support. Hence $\mu_0 \odot F_0 \leq 1 - \epsilon_0$ and $M(\mu_0 \odot F_0) < 1$ for any mean M, which contradicts (b). If X is topological left introverted, (b) clearly implies (c). Finally, if (c) holds but not (a), then the same argument shows that

$$Q(\mu_0) \odot M(F_0) = M(\mu_0 \odot F_0) < 1,$$

which is again a contradiction.

REMARKS. Note that we do not require X to be a lattice, unlike in [**4**]. (A norm closed subalgebra of $B(S)$ must be a lattice, see Douglas [**3**].)

3. Main results. Suppose now X is topological left introverted. Let $S_1 = M(X)$ be the semigroup of means on X with Arens product and let $M_T(X) = \{M \in M(X)\colon M(F) = 1 \text{ for all } \chi_T \leq F \leq 1,\ F \in X\}$. The following theorem gives an interesting relationship between topological left thickness and left thickness (as defined in [**4**]).

THEOREM 3.1. *Let X be topological left introverted. Then the following statements are equivalent:*

(a) *T is topological left X-thick.*

(b) *$M_T(X)$ contains a left ideal of $M(X)$.*

(c) *$M_T(X)$ is left $B(S_1)$-thick in S_1.*

PROOF. Suppose T is topologically left thick. By Theorem 2.1 (c), there is a mean M on X such that $N \odot M(F) = 1$ for all mean N on X and all $\chi_T \leq F \leq 1$, $F \in X$. This means that $M_T(X)$ contains the left ideal $M(X) \odot M$ of

$M(X)$. Hence (a) implies (b) which clearly implies (c), since any left ideal in S_1 is certainly left $B(S_1)$-thick. ([**4**, Remark 5, p. 39]. See also Remark 3 below.) Finally suppose $M_T(X)$ is left $B(S_1)$-thick in S_1. Consider the map $\tau\colon X \to B(S_1)$ defined by $\tau(F)(M) = M(F)$, $F \in X$, $M \in S_1$. It is straightforward to show that τ is bounded linear, isometric and order preserving. Moreover, $\tau(1) = 1$ and τ commutes with "left translations". That is, $\tau(\mu \odot F) = L(Q(\mu))\tau F$ where $L(M)$ is the left translation operator in $B(S_1)$ defined by $(L(M)h)(N) = h(M \odot N)$, M, $N \in S_1$, $h \in B(S_1)$. By [**4**, Theorem 1], there is some mean θ on $B(S_1)$ such that $\theta(L(N)h) = 1$ for all $N \in S_1$ and all $h \in B(S_1)$ with $\xi_{M_T(X)} \leq h \leq 1$. Then $M = \tau^*\theta$ is a mean on X. Moreover, if $F \in X$, $\chi_T \leq F \leq 1$, then $\xi_{M_T(X)} \leq \tau(F) \leq 1$ and so for any $\mu \in M_0(S)$,

$$M(\mu \odot F) = \tau^*\theta(\mu \odot F) = \theta(\tau(\mu \odot F)) = \theta(L(Q(\mu))\tau F) = 1$$

since $Q(\mu) \in S_1$. By Theorem 2.1, T is topological left X-thick. This completes the proof.

REMARKS. (1) Even when S is discrete, Theorem 3.1 is new and more general than Junghenn [**4**, Theorem 1] and different from Wong [**13**, Theorem 3.3]. For in this case, using the notations in [**4** and **13**] with $F = X$, a left introverted, left invariant linear subspace of $B(S)$ containing 1 (but not necessarily a norm closed subalgebra), we have

$$M(X) = \mathrm{ClCo}\, e(S) \quad \text{and} \quad M_T(X) \subset \mathrm{ClCo}\, e(T)$$

(equality holds if $\xi_T \in X$). Here $\mathrm{ClCo}\, e(T)$ denotes the $\sigma(X^*, X)$-closure of the convex hull of $e(T)$. Of course, under these conditions, the statements of Theorem 3.1 and those in [**13**, Theorem 3.3] are all equivalent as well.

(2) The map τ in general is not onto, unlike the Gel′fand Naĭmark isomorphism used in [**4** and **13**].

(3) If T is or contains a left ideal of S, then T is topological left X-thick for any X in $M(S)^*$ and left F-thick for any F in $B(S)$. This can be easily verified from the definitions. In general, a left thick subset need not contain a left ideal. An exception is of course the set $M_T(X)$ in S_1.

The full algebra $B(S_1)$ can also be replaced by some smaller subspace as in the following:

THEOREM 3.2. *If X is topological left introverted and W is any left invariant linear sublattice of $B(S_1)$ containing $\tau(X)$, then each of the statements in the preceding theorem is also equivalent to*

(d) *$M_T(X)$ is left W-thick in S_1.*

PROOF. The proof proceeds as before, using Remark 3 above for (b) implies (d) and Wong [**13**, Theorem 3.1] (in place of Junghenn [**4**, Theorem 1]) for (d) implies (a) since W need not be left introverted. (Again, recall that any norm closed subalgebra of $B(S_1)$ is always a lattice.)

4. The extent of $\tau(X)$. In general, the smaller the space W, the easier for $M_T(X)$ to be left W-thick (and T topological left X-thick). From Theorem 3.2, the smallest W is the left invariant linear sublattice $W(X)$ of $B(S_1)$ generated by $\tau(X)$. But this is not readily identifiable. We now give some easily identifiable choices of W.

EXAMPLES 4.1. Let S_1 have the weak topology of X^*, then S_1 is a topological semigroup and $\tau(X) \subset C(S_1)$, the continuous functions in $B(S_1)$. We can therefore take W to be $C(S_1)$.

Note that the space $C(S_1)$ is a left invariant norm closed linear sublattice of $B(S_1)$. In general, $C(S_1)$ need not be left introverted.

EXAMPLE 4.2. Let S_1 have the norm topology of X^*, then S_1 is a (jointly continuous) topological semigroup and $\tau(X) \subset UC(S_1)$, the uniformly continuous functions in $B(S_1)$. For if $M, N, P \in S_1$, $F \in X$, we have

$$\begin{aligned} \|L(M)\tau(F) - L(N)\tau(F)\| &= \sup_P |\tau(F)(M \odot P) - \tau(F)(N \odot P)| \\ &\leq \|F\| \cdot \|M - N\|. \end{aligned}$$

Similarly, $\|R(M)\tau(F) - R(N)\tau(F)\| \leq \|F\| \cdot \|m - N\|$.

Note that $UC(S_1)$ is a left introverted, left invariant norm closed linear sublattice of $C(S_1)$. Hence we can take $W = UC(S_1)$.

EXAMPLE 4.3. Let X be the space of all weakly almost periodic functionals F in $M(S)^*$ (i.e. the weak closure of $\{F \odot \mu \colon \mu \in M_0(S)\}$ is weakly compact in $M(S)^*$ where $F \odot \mu(\nu) = F(\nu * \mu)$, $\nu \in M(S)$) and let S_1 have the weak* topology of X^*. It is easy to show that X is a norm closed topological left introverted, topological left invariant linear subspace of $M(S)^*$ containing 1. As in Kharaghani [**5**, Lemmas 3.1 and 3.2, p. 44], one can show that the weak closure of $\{F \odot \mu \colon \mu \in M_0(S)\} = \{M_L(F) \colon M \in S_1\}$ and as a result, S_1 is a compact topological semigroup with the weak* topology. Clearly $\tau(X) \subset C(S_1)$.

REMARKS. (1) In general, S_1 need not be a topological semigroup in weak* topology as multiplication may not be continuous in the second variable.

(2) One can also show that right and left weak almost periodicity are equivalent by using the corresponding result in $C(S_1)$ via the isometry τ.

(3) In general, $\tau(X)$ is only left $M_0(S)$-invariant but not left invariant in $B(S_1)$. However, if X is the weakly almost periodic functionals, then $\tau(X)$ is left invariant. For if $M \in S_1$ and $\mu_\alpha \in M_0(S)$ are such that $Q(\mu_\alpha) \to M$ in S_1, then $Q(\mu_\alpha) \odot N(F) \to M \odot N(F)$ for any $N \in S_1$, $F \in X$. Thus $L(Q(\mu_\alpha))\tau(F) \to L(M)\tau(F)$ pointwise on S_1, hence weakly in $C(S_1)$. Therefore $L(M)\tau(F) \in \tau(X)$ (which is weakly closed in $C(S_1)$). Now if W is the norm closed subalgebra (or sublattice) of $C(S_1)$ generated by $\tau(X)$, then W is automatically left invariant (since the left translation operators in $C(S_1)$ (or $B(S_1)$) are multiplicative, linear and norm continuous) and Theorem 3.2 continues to hold for this W.

5. An application. We now present an application to left amenable semigroups. But first a lemma. The reader is referred to Wong [**12**, Lemma 3.1 and Theorem 4.1] for a proof.

LEMMA 5.1. *Let T be a locally compact Borel subsemigroup of a locally compact semigroup S. Then*

(1) *$M(T)$ is isometrically isomorphic to the algebra of all measures $\mu \in M(S)$ with $|\mu|(T') = 0$ under the mapping $\mu \to \mu_T$, the restriction of $\mu \in M(S)$ to Borel sets in T.*

(2) *Each $F \in M(S)^*$ induces $F|_T \in M(T)^*$ such that $F|_T(\mu_T) = F(\mu)$, $\mu \in M(S)$, $|\mu|(T') = 0$. The map $F \to F|_T$ is bounded linear and order preserving.*

Moreover $1|_T = 1$ *and*

$$\mu_T \odot F|_T = (\mu \odot F)|_T(M), \qquad \mu \in M(S),\ |\mu|(T') = 0.$$

(3) *If* M_0 *is a mean on* $M(T)^*$ *and* $M(F) = M_0(F_T)$, $F \in M(S)^*$, *then* M *is a mean on* $M(S)^*$. *Moreover* $(M_0)_L(F|_T) = M_L(F)|_T$.

The next theorem is a topological variation of Junghenn [**4**, Corollary to Theorem 2]. It is also an extension of a result in Wong [**11**, Theorem 3.4, p. 578]. The earliest form of these results is due to Mitchell [**6**]. Our proof is different from that in [**4**] which depends on the Gel'fand Naĭmark isomorphism (our map τ is not onto).

THEOREM 5.2. *If* X *is a topological left introverted, topological left invariant linear sublattice of* $M(S)^*$ *containing* 1 *and* T *is a topological left* X*-thick subsemigroup of* S, *then* X *is topological left amenable iff* $X|_T = \{F|_T \colon F \in X\}$ *is.*

PROOF. The preceding lemma shows that $X|_T$ is a topological left introverted, topological left invariant linear subspace of $M(T)^*$ containing 1. Suppose X is topological left amenable. Since T is topological left X-thick, by Theorem 2.1 (c), there is a topological left invariant mean (TLIM) M in $M_T(X)$ (note that $N \odot M$ is a TLIM whenever N is). Put $M_0(F|_T) = M(F)$, $F \in X$. The above lemma shows that M_0 is a TLIM on $X|_T$ provided that (1) M_0 is well defined, (2) $M(F) \geq 0$ whenever $F|_T \geq 0$ and (3) $M(F) = 1$ if $F|_T = 1$. To show (2), take $F \in X$, $\|F\| = 1$ and $F|_T \geq 0$ (as in [**4**]). Then $(1+F)(\mu) \geq \chi_T(\mu) = \mu(T)$ for any $\mu \in M_0(S)$. (This is clear if either $\mu(T) = 0$ or $\mu(T) = 1$. In general, write $\mu = \mu(T)\nu + \mu(T')\theta$ where $\nu, \theta \in M_0(S)$, $\nu(T) = 1$, $\theta(T) = 0$.) Hence $1 + F \geq \chi_T$ and $\chi_T \leq G = 1 \wedge (1+F) \leq 1$. Since $M \in M_T(X)$, $M(1+F) \geq M(G) = 1$ and $M(F) \geq 0$. (This is the only place we use the fact that X is a lattice.) It follows that $M(F) = 0$ whenever $F|_T = 0$ which implies (1) of which (3) is a special case.

Conversely, assume $X|_T$ has a TLIM M_0. Define M_1 on X by $M_1(F) = M_0(F|_T)$, $F \in X$. The above lemma shows that M_1 is a topological left T-invariant mean on X (i.e. $M_1(\mu \odot F) = M_1(F)$ for all $F \in X$ and all $\mu \in M_0(S)$ such that $\mu(T') = 0$). Then $M \odot M_1 = M_1$ for any $M \in M_T(X)$. For if $\mu \in M_0(S)$ with $\mu(T') = 0$ and $F \in X$, we have $(M_1)_L(F)(\mu) = M_1(\mu \odot F) = M_1(F) \cdot 1(\mu)$ or $(M_1)_L(F) = M_1(F) \cdot 1$ on T, which by (1) above implies that $M((M_1)_L(F)) = M_1(F)$, $F \in X$. That is $M \odot M_1 = M_1$. Now by Theorem 3.2, $M_T(X)$ contains a left ideal I. Take any $N \in I$. Then for any $\mu \in M_0(S)$, $Q(\mu) \odot M_1 = Q(\mu) \odot (N \odot M_1) = (Q(\mu) \odot N) \odot M_1 = M_1$ and M_1 is a topological left invariant mean on X. This completes the proof.

REMARK. For sufficiency, we do not require X to be a lattice.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CALGARY, CALGARY T2N 1N4, ALBERTA, CANADA