PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 3, November 1982

GENERIC EXISTENCE OF A SOLUTION FOR A DIFFERENTIAL EQUATION IN A SCALE OF BANACH SPACES

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ABSTRACT. Let $\{X_s : \alpha \leq s \leq \beta\}$ be a scale of Banach spaces, J a real interval, U an open subset of $J \times X_s$ for some s. In this paper we prove that the existence of solutions for

$$x' = A(t)x + f(t,x), \quad x(t_0) = x_0,$$

is a generic property, when $A(t)$ is an operator satisfying

$$|A(t)|_{L(X_{s'};X_s)} \leq M(s'-s)^{-1} \quad (M > 0 \text{ independent of } s, s', t)$$

in the scale $\{X_s\}$ and $f : J \times U \to X_\beta$ is continuous.

1. Introduction. A property is said to be generic in a Baire space E if it holds in a residual subset of E. Let X be an infinite dimensional Banach space, $\mathbf{R}$ the set of real numbers and V an open subset of $\mathbf{R} \times X$. Denote by $C(V;X)$ the set of all continuous mappings from V into X, endowed with the topology of uniform convergence. Lasota and Yorke [**7**] proved that the existence of solutions for the differential equation

$$\text{(I)} \qquad x' = f(t,x), \quad x(t_0) = x_0,$$

is a generic property in $C(V;X)$. The generic existence of solutions is also studied in [**12, 1, 3**] for ordinary differential equations in Banach spaces, and in [**2, 4, 10**] for integral and functional equations.

In this paper we study the corresponding problem for differential equations in a scale of Banach spaces. These equations were introduced by Ovcyannikov [**8**] who proved the existence of solutions for the linear differential equation

$$x' = A(t)x, \quad x(t_0) = x_0,$$

where $A(t)$ is a linear operator satisfying conditon (1) (see §2) in a scale $\{X_s : \alpha \leq s \leq \beta\}$ of Banach spaces. Treves [**11**] considered the equation

$$\text{(II)} \qquad x' = A(t)x + g(t), \quad x(t_0) = x_0,$$

where $g : J \to X_\beta$ is a continuous mapping on a real interval J, and he applied this equation to solve a Cauchy-Kowaleska problem. Deimling [**5**, p. 26] proved the existence of solutions for the more general equation

$$\text{(III)} \qquad x' = A(t)x + f(t,x), \quad x(t_0) = x_0,$$

Received by the editors January 19, 1981 and, in revised form, November 2, 1981.
1980 *Mathematics Subject Classification*. Primary 34G20; Secondary 54C50.
Key words and phrases. Differential equation, scale of Banach spaces, Baire category, residual subset, generic property.

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where $f\colon J \times D \to X_\beta$ is a bounded, uniformly continuous mapping, α-Lipschitz with respect to the second variable, and D denotes a closed ball in X_s for some fixed s in (α, β). Let U be an open subset of $J \times X_s$ and denote by $C(U; X_\beta)$ the set of all continuous mappings from U in X_β endowed with the topology of uniform convergence. In this paper (Theorem 8) we prove that problem (III) has a unique solution for every function f in a residual subset of the Baire space $C(U; X_\beta)$. In order to better clarify the significance of this theorem, notice that the subset of $C(U; X_\beta)$ formed by all locally α-Lipschitz mappings is of Baire first category [**7**, Theorem 2] and that for the easier problem (I) there exists a dense subset $\mathcal{E}$ of $C(V; X)$ such that (I) has no solution for every f in $\mathcal{E}$ [**9**].

2. Minimal interval of existence of solution and continuous dependence. In order to prove the generic existence of solutions we need to know some fundamental properties of the solutions of (III). In the following we will assume that $\{x_s\colon \alpha \leq s \leq \beta\}$ is a scale of Banach spaces such that $X_{s'} \subset X_s$ and $|x|_s \leq |x|_{s'}$ for $s < s'$ and $x \in X_{s'}$, where $|x|_s$ denotes norm in X_s. For some fixed positive number a, we will denote by J the interval $(t_0 - a, t_0 + a)$, t_0 being an arbitrary real number. If X is a metric space, x is in X, and A is a subset of X, we denote by $d(x, A)$ the number $\inf\{d(x,y) : y \in A\}$ and by $B(x,r)$ ($\overline{B}(x,r)$) the open (closed) ball centered at x with radius r. Finally if X, Y are metric spaces and Y is complete, $C(X;Y)$ will denote the complete metric space of all continuous mappings from X into Y with the metric

$$d(f,g) = \sup\{\min\{1, d(f(x), g(x))\}\colon x \in X\}.$$

LEMMA 1 [**5**, THEOREM 1.2]. *Let $\{X_s\colon \alpha \leq s \leq \beta\}$ be a scale of Banach spaces, $g\colon J \to X_\beta$ continuous, $x_0 \in X_\beta$, $A\colon J \to L(X_{s'}; X_s)$ a continuous mapping for every pair (s', s), $\alpha \leq s < s' \leq \beta$, satisfying*

$$(1) \qquad |A(t)|_{L(X_{s'};X_s)} \leq M(s'-s)^{-1} \quad (M > 0 \textit{ independent of } s, s', t).$$

Then for every $s \in [\alpha, \beta)$ problem (II) *has a unique solution $\phi\colon (t_0 - \delta, t_0 + \delta) \to X_s$ where $\delta = \min\{a, (\beta - s)/Me\}$ and*

$$(2) \qquad |\phi(t) - x_0|_s \leq \left(|x_0|_\beta + (\beta - s)M^{-1} \max_{[t_0,t]\, or\, [t,t_0]} |g(\tau)|_\beta\right) \times Me|t - t_0|(\beta - s - Me|t - t_0|)^{-1}.$$

Let $A(t)$ be as in Lemma 1, Ω an open subset of X_s, x_0 a point in $\Omega \cap X_\beta$, $f\colon J \times \Omega \to X_\beta$ continuous and ϕ a solution of (III) defined on an interval (τ_1, τ_2) contained in J. We recall [**7**] that ϕ is said to be unlimited if there does not exist $\lim(t, \phi(t))$ as $t \to \tau_i$ $(i = 1, 2)$ in $J \times \Omega$.

THEOREM 2. *Let $A(t)$, x_0 be as in Lemma* 1, *Ω an open subset of X_s which contains x_0, $f\colon J \times \Omega \to X_\beta$ continuous and bounded, and c a bound for f.*

(a) *If ϕ is a solution of* (III) *defined on $l = [t_0 - b, t_0 + b]$ for some $b < \min\{a, (\beta - s)/Me\}$, then $\phi(t)$ is in $X_{s'}$ for every t in l and $s' \in [s, \beta - Meb)$. Furthermore one has*

$$(3) \quad |\phi(t) - x_0|_{s'} \leq N(s', b, x_0) = (|x_0|_\beta + c(\beta - s)M^{-1})Meb(\beta - s' - Meb)^{-1},$$

and for every $\epsilon < \beta - s' - Meb$ one has

$$(4) \qquad |\phi(t) - \phi(\bar{t})|_{s'} \leq P(s', \epsilon, b, x_0)|t - \bar{t}| \qquad (t, \bar{t} \in l),$$

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where $P(s', \epsilon, b, x_0) = (|x_0|_\beta + N(s'+\epsilon, b, x_0))(M\epsilon^{-1} + c)$.

(b) *If* ψ *is an unlimited solution of* (III), *then* ψ *is defined at least on the interval* $(t_0 - \delta, t_0 + \delta)$, $\delta = \min\{a, (\beta - s)/Mhe\}$ *where*

$$h = \begin{cases} 1 & \text{if } d = d(x_0, \Omega^c) = +\infty, \\ (|x_0|_\beta + cM^{-1}(\beta - s) + d)d^{-1} & \text{otherwise.} \end{cases}$$

PROOF. (a) Define $g(t) = f(t, \phi(t))$, $t \in l$. From Lemma 1 we know that for every $s' \in [s, \beta - Meb)$ problem (II) has a unique solution $\overline{\phi}$: $(t_0 - \delta', t_0 + \delta') \to X_{s'}$ where $\delta' = \min\{a, (\beta - s')/Me\} > b$. Since ϕ and $\overline{\phi}$ are solutions of (II) in X_s we derive from Lemma 1 that $\phi(t) = \overline{\phi}(t)$ for every t in $(t_0 - \delta', t_0 + \delta')$. Thus $\phi(t)$ belongs to $X_{s'}$ for every $s' \in [s, \beta - Meb)$. Inequality (3) follows quite immediately from (2). Condition (1) for $(s', s' + \epsilon)$, applied to the integral equation for (III), implies (4).

(b) Assume ψ is defined on (τ_1, τ_2) where $\tau = \min\{|t_0 - \tau_1|, |t_0 - \tau_2|\} < \delta$. Since ψ is defined on $(t_0 - \tau, t_0 + \tau)$ we obtain from (4)

$$|\psi(t) - \psi(\bar{t})|_s \le P(s + \epsilon, \epsilon, \tau, x_0)|t - \bar{t}| \quad (t, \bar{t} \in (t_0 - \tau, t_0 + \tau))$$

for some $\epsilon < \beta - s - Me\tau$, i.e. ψ satisfies a Cauchy condition. Hence there exists $\lim \psi(t_0 + t) = \rho$ as $t \to \tau$ or $t \to -\tau$, and we claim that ρ is in Ω. Indeed, from (3) we derive $|\rho - x_0|_s \le N(s, \tau, x_0)$. Using $\beta - s \ge h\delta Me > h\tau Me$ and the definition of h for $d < +\infty$ we have

$$|\rho - x_0|_s < (|x_0|_\beta + cM^{-1}(\beta - s))(h-1)^{-1} = d.$$

LEMMA 3. *Let* $A(t)$, x_0 *be as in Lemma* 1 *and assume the following hold:*

(i) $D = \overline{B}(x_0, R)$ *is a closed ball in* X_s *for some fixed* $s \in (\alpha, \beta)$.

(ii) $f_0 : J \times D \to X_\beta$ *is continuous, bounded and Lipschitz-continuous with respect to the second variable with modulus* L.

(iii) $\{(t_n, x_n)\}$ *is a sequence which converges to* (t_0, x_0) *in* $J \times X_\beta$.

(iv) $\{f_n\}$ *is a sequence which converges to* f_0 *in* $C(J \times D; X_\beta)$.

(v) *For every nonnegative integer* n, ϕ_n *is an unlimited solution of*

$$\text{(IV)} \qquad x' = A(t)x + f_n(t, x), \quad x(t_n) = x_n.$$

Let c *be any real number greater than* $\sup\{|f(t,x)|_\beta : (t,x) \in J \times D\}$ *and set* $h = 2(|x_0|_\beta + R + c(\beta - s)M^{-1})R^{-1}$. *Then there exists a positive integer* n_0 *such that for every* γ, $0 < \gamma \le \min\{a/4, (\beta - s)/4hMe, (4Le)^{-1}\}$ *and for every* $n \ge n_0$, ϕ_n *is defined from* $l = [t_0 - \gamma, t_0 + \gamma]$ *into* $X_{s'}$ *and* $\{\phi_n\} \to \phi_0$ *in* $c(l; X_{s'})$ *for every* $s', s \le s' < \beta - 3Me\gamma$.

PROOF. Assume n large enough so that $|f_n(t,x)|_\beta \le c$ and $|x_n - x_0|_\beta < R/2$. Since $|x_n|_\beta < |x_0|_\beta + 2^{-1}R$ and $d(x_n, D^c) > 2^{-1}R$ we obtain from Theorem 2(b) that ϕ_n is defined on $J \cap (t_n - 4\gamma, t_n + 4\gamma)$. Hence for n large enough, ϕ_n is defined on l. Set $\gamma' = \gamma + |t_n - t_0|$. Since l is contained in $[t_n - \gamma', t_n + \gamma']$ we deduce from Theorem 2(a) that $\phi_n(t)$ belongs to $X_{s'}$ $(t \in l)$ for every $s' < \beta - Me\gamma'$. Thus for n large enough so that $|t_n - t_0| < \gamma$, say $n \ge n_0$, we have that $\phi_n(t)$ belongs to $X_{s'}$ $(t \in l, n \ge n_0)$ for every $s' < \beta - 2Me\gamma$. Denote by $D_{s'}$ the closed set $D \cap X_{s'}$ in $X_{s'}$ and define for every u in $C(l; D_{s'})$ a mapping $T_n u$ $(n \ge n_0)$ by: $T_n u$ is the unique solution on l of

$$\text{(V)} \qquad x' = A(t)x + f_n(t, u(t)), \quad x(t_n) = x_n.$$

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Notice that Lemma 1 assures the existence and uniqueness of $T_n u$. Furthermore by Theorem 2(a), $T_n u(t)$ belongs to $X_{s'}$ for every t in l and $s' < \beta - 2Me\gamma < \beta - Me\gamma'$.

CLAIM 1. $T_n u$ belongs to $C(l; D_{s'})$ for every u in $C(l; D_{s'})$, $s \leq s' < \beta - 2Me\gamma$, $n \geq n_0$.

Choose $\epsilon < \beta - s' - 2Me\gamma < \beta - s' - Me\gamma'$. From (4) we have

$$|T_n u(t) - T_n u(\bar{t})|_{s'} \leq P(s', \epsilon, \gamma', x_n)|t - \bar{t}|.$$

Thus $T_n u$ is a continuous mapping. Hence it suffices to prove that $T_n u(t)$ is in D for every $t \in l$. From (3) we have

$$|T_n u(t) - x_n|_s \leq N(s, \gamma', x_n) < N(s, 2\gamma, x_n).$$

Using $|x_n|_\beta < |x_0|_\beta + R/2$, $\beta - s \geq 2hMe\gamma$, the definition of h, and following an argument as in the final part of Theorem 2 we derive $N(s, 2\gamma, x_n) < R/2$. Hence $|T_n u(t) - x_0|_\beta < R$.

CLAIM 2. For u in $C(l; D)$ denote $||u|| = \sup\{|u(t)|_s : t \in l\}$. Then for every s', $s \leq s' < \beta - 2Me\gamma$, u, v in $C(l; D_{s'})$ and $t \in l$ we have $|T_0 u(t) - T_0 v(t)|_{s'} \leq 2^{-1}||u - v||$.

Since $T_0 u - T_0 v$ is a solution of the equation

$$x' = A(t)x + f_0(t, u(t)) - f_0(t, v(t)), \quad x(t_0) = x_0,$$

we obtain from (2)

$$|T_0 u(t) - T_0 v(t)|_{s'} \leq (\beta - s')M^{-1} L \max_{\tau \in l} |u(\tau) - v(\tau)|_s Me\gamma(\beta - s' - Me\gamma)^{-1}$$

that is less than $2^{-1}||u - v||$. Here we use $-Me\gamma \geq -(\beta - s')/2$ and the definition of γ.

CLAIM 3. For every $s, s \leq s' < \beta - 3Me\gamma$ the sequence $\{T_n\}$ converges to T_0 uniformly on $C(l; D_{s'})$.

Let s' satisfy $s \leq s' < \beta - 3Me\gamma$ and choose $s'' < \beta - 2Me\gamma$ such that $s'' - s' > Me\gamma$. Since for every u in $C(l; D_{s'})$, $T_n u - T_0 u$ is the solution of the problem

$$x' = A(t)x + f_n(t, u(t)) - f_0(t, u(t));$$
$$x(t_0) = y_n = x_n - x_0 + \int_{t_n}^{t_0} [A(\tau) T_n u(\tau) + f_n(\tau, u(\tau))] \, d\tau$$

in the scale $\{X_r : \alpha \leq r \leq s''\}$ (notice that y_n belongs to $X_{s''}$) we obtain from (2),

$$\begin{aligned} &|T_n u(t) - T_0 u(t)|_{s'} \\ &\quad \leq |y_n|_{s'} + \left(|y_n|_{s''} + M^{-1}(s'' - s') \max_{t \in l} |f_0(t, u(t)) - f_n(t, u(t))|_{s''} \right) \\ &\quad\quad \times Me\gamma(s'' - s' - Me\gamma)^{-1}. \end{aligned}$$

Since $\{f_n\} \to f_0$ uniformly, it suffices to prove that $|y_n|_{s''} \to 0$ uniformly with respect to u in $C(l; D_{s'})$. From (3) there exists a constant N such that $|T_n u(t)|_{s'' + \epsilon} < N$ for some small $\epsilon > 0$ and for every u in $C(l; D_{s'})$, $t \in l$. Hence condition (1) applied to $(s'', s'' + \epsilon)$ implies

$$|y_n|_{s''} \leq |x_n - x_0|_{s''} + |t_n - t_0|(MN\epsilon^{-1} + c)$$

and the right-hand side of this inequality converges to 0 and is independent of u.

To complete the proof of the lemma, fix s', $s \leq s' < \beta - 3Me\gamma$. Let η be an arbitrary positive number and n_0 large enough so that $|T_n u(t) - T_0 u(t)|_{s'} < \eta/2$ for

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every $n \geq n_0$, $t \in l$ and $u \in C(l; D_{s'})$. Since ϕ_n is a fixed point of T_n we have

$$\begin{aligned}\sup_{t\in l} |\phi_n(t) - \phi_0(t)|_{s'} = \sup_{t\in l} |T_n\phi_n(t) - T_0\phi_0(t)|_{s'} &\leq 2^{-1}\|\phi_n - \phi_0\| + \eta/2 \\ &\leq 2^{-1} \sup_{t\in l} |\phi_n(t) - \phi_0(t)|_{s'} + \eta/2.\end{aligned}$$

Therefore $\{\phi_n\} \to \phi_0$ in $C(l; D_{s'})$.

THEOREM 4. *Let $A(t)$ be as in Lemma* 1, *and assume the following hold*:

(i) *Ω is an open subset of X_s for some fixed $s \in (\alpha, \beta)$.*

(ii) *$f_0: J \times \Omega \to X_\beta$ is locally Lipschitz-continuous and bounded.*

(iii) *$\{f_n\}$ is a sequence which converges to f_0 in $C(J \times \Omega, X_\beta)$.*

(iv) *$\{(t_n, x_n)\}$ is a sequence which converges to (t_0, x_0) in $J \times X_\beta$.*

(v) *For any nonnegative integer n, ϕ_n is an unlimited solution of* (IV).

Then, for every real number $b < \delta$ where $\delta = \min\{a, (\beta - s)/Mhe\}$, h defined as in Theorem 2, one has: There exists a positive integer n_0 such that for every $n \geq n_0$, ϕ_n is defined on $l = [t_0 - b, t_0 + b]$ and $\{\phi_n\}$ converges to ϕ_0 in $C(l; X_{s'})$ for every s', $s \leq s' < \beta - Meb$.

PROOF. From Theorem 2(b), ϕ_0 and ϕ_n are defined on l for n large enough. Using Theorem 2(a) it is easy to prove that $\phi_0(t)$ and $\phi_n(t)$ belong to $X_{s'}$ for every $t \in l$, $s' \in [s, \beta - Meb)$ and n large enough, say $n \geq n_0$. Let s_1, s_2 be real numbers, $s \leq s_1 < s_2 < \beta - Meb$. It is easy to prove from (4) that there exist a constant P, independent of n, such that

$$(5) \qquad |\phi_n(t) - \phi_n(\bar{t})|_{s_2} \leq P|t - \bar{t}|, \quad t, \bar{t} \in l, \quad n \geq n_0.$$

Define the set $S = \{t \leq b: \{\phi_n\} \to \phi_0 \text{ in } C([t_0 - t, t_0 + t]; X_{s'}) \text{ for some } s',\ s_1 < s' \leq s_2\}$. Since zero is in S we have $S \neq \varnothing$. Furthermore, the equicontinuity condition (5) implies that S is a closed set. Set $\sigma = \sup S$. Then $\{\phi_n\} \to \phi_0$ in $C([t_0 - \sigma, t_0 + \sigma]; X_{s'})$ for some s', $s_1 < s' \leq s_2$. We claim that $\sigma = b$. Otherwise set $\Omega_1 = \Omega \cap X_{s_1}$ and notice that $f_0: J \times \Omega_1 \to X_\beta$ satisfied the hypotheses of Lemma 3 in a neighborhood of $(t_0 + \sigma, \phi_0(t_0 + \sigma))$ for the sequences $\{f_n\} \to f_0$ in $C(J \times \Omega_1; X_\beta)$ and $\{(t_0 + \sigma, \phi_n(t_0 + \sigma))\}$ converging to $(t_0 + \sigma, \phi_0(t_0 + \sigma))$ in $X_{s'}$. Therefore we can apply Lemma 3 (where s is replaced by s_1, and β by s') to prove that $\{\phi_n\} \to \phi_0$ in $C([t_0 + \sigma - \gamma, t_0 + \sigma + \gamma]; X_{s''})$ for some $\gamma > 0$ and s'', $s_1 < s'' < s'$. Since the same argument holds for the point $(t_0 - \sigma, \phi_0(t_0 - \sigma))$ we get a contradiction. Thus $\{\phi_n\} \to \phi_0$ in $C(l; X_{s'})$ for some $s' > s_1$ and "a fortiori" in $C(l; X_{s_1})$.

3. Generic existence of solutions. To prove the generic existence of solutions of (3) we follow the pattern developed in [**7**] for ordinary differential equations. In the following U will denote a subset of $\mathbf{R} \times X_s$ for some fixed $s \in (\alpha, \beta)$, $u_0 = (t_0, x_0)$ a point in U ($x_0 \in X_\beta$) and X the Baire space $C(U; X_\beta)$. If $u = (t, x)$ and $\bar{u} = (\bar{t}, \bar{x})$ are in U $d(u, \bar{u})$ will mean $\max\{|x - \bar{x}|_s, |t - \bar{t}|\}$. For some number k, $0 < k < 1$, choose a number $\bar{s}$ in $(s, ks + (1-k)\beta)$. Let f be a mapping in X, $c'_f = |f(u_0)|_\beta + 1$, $U_f = \operatorname{int}\{(t, x) \in U: |f(t, x)|_\beta \leq c'_f\}$, $c_f = c'_f + 1$, $D_f = d(u_0, U_f^c)$ and $\delta_f = \min\{D_f, (\beta - s)/Meh_f\}$ where

$$h_f = \begin{cases} 1 & \text{if } D_f = +\infty, \\ (|x_0|_\beta + c_f M^{-1}(\beta - s) + D_f)D_f^{-1} & \text{if } D_f < +\infty. \end{cases}$$

Denote by K_f the interval $(t_0 - \delta_f, t_0 + \delta_f)$ and by J_f the interval $[t_0 - k\delta_f, t_0 + k\delta_f]$. Notice that $\bar{s}$ is in $(s, \beta - Mek\delta_f)$ for every f in X.

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Let f_1, f_2 be mappings in X; $u_i = (t_i, x_i)$ $(1 = 1,2)$ points in $J \times X_\beta$. If ϕ_i $(i = 1,2)$ are solutions in $X_{\overline{s}}$ of

$$\text{(VI)} \qquad x' = A(t)x + f_i(t,x), \quad x(t_i) = x_i,$$

defined on J_f we denote

$$d_f(\phi_1, \phi_2) = \sup\{|\phi_1(t) - \phi_2(t)|_{\overline{s}} : t \in J_f\},$$

$$\mu(f,\delta) = \sup\{d_f(\phi_1,\phi_2) \colon f_1 \in B(f,\delta); u_i \in B(u_0,\delta),$$
$$\phi_i \text{unlimited solutions of (VI) defined on} J_f\},$$
$$V(f) = \limsup \mu(f,\delta) \quad \text{as } \delta \to 0.$$

LEMMA 5. *Let f be a mapping in X such that $V(f) = 0$. Then*
(i) *there exists a solution $\phi \colon J_f \to X_s$ of* (III);
(ii) *if $\overline{\phi}$ is another solution of* (III) *defined on J_f, then $\phi(t) = \overline{\phi}(t)$ for every t in J_f;*
(iii) *the solution ϕ depends continuously on f and u; i.e. if $\{f_n\} \to f$ in X, $\{u_n\} \to u_0$ in $J \times X_\beta$ and ϕ_n is an unlimited solution of* (IV) *, then there exist a positive integer n_0, such that ϕ_n is defined on J_f if $n \geq n_0$ and $\{\phi_n\} \to \phi$ in $C(J_f; X_s)$.*

PROOF. From [7, Lemma 1] we can choose a sequence of locally Liptschitz mappings $\{f_n\}$ which converges to f in X assuming without loss of generality $|f_n(t,x)|_\beta < c_f$ on U_f. From Theorem 2 the unique unlimited solution ψ_n of (IV) (whose existence is given by [5, Theorem 2.4] is defined from J_f into $X_{s'}$, $s' \in [s, \beta - Mek\delta_f)$ for n large enough. Since $V(f) = 0$ we derive that $\{\psi_n\}$ is a Cauchy sequence in $C(J_f; X_{\overline{s}})$ and thus $\{\psi_n\}$ converges in $C(J_f; X_{\overline{s}})$ to a mapping ϕ. Using the integral equation for (IV) we obtain that ϕ is a solution of (III) in X_s. (Recall that $A(t)$ is a continuous operator from $X_{\overline{s}}$ into X_s.) Since $V(f) = 0$ it is easy to prove (ii). In order to prove (iii) notice that ϕ_n is defined on J_f for large n. Then $V(f) = 0$ implies that $\{\phi_n\} \to \phi$ in $C(J_f; X_{\overline{s}})$ and "a fortiori" in $C(J_f; X_s)$.

The next lemma follows immediately from Theorem 4.

LEMMA 6. *Let f be a locally Lipschitz mapping in X. Then $V(f) = 0$.*

LEMMA 7. *$V \colon X \to \mathbf{R}^+$ is continuous on the set $V^{-1}(\{0\})$.*

PROOF. Assume $V(f) = 0$. If V is not continuous at f, there exists a positive number η and a sequence $\{f_n\} \to f$ such that $V(f_n) > \eta$. For each n there exists a sequence of positive numbers $\delta_{mn} \to 0$, $\delta_{mn} < 1/m$, such that $\mu(f_n, \delta_{mn}) > \eta/2$. Hence there are sequences $\{\phi_{imn}\}$, $\{f_{imn}\}$, $\{u_{imn}\}$ $(i = 1,2)$ such that ϕ_{imn} is an unlimited solution of $x' = A(t)x + f_{imn}(t,x)$, $x(t_{imn}) = x_{imn}$ $(i = 1,2)$ defined on J_{f_n}, f_{imn} is in $B(f_n, \delta_{mn})$, u_{imn} is in $B(u_n, \delta_{mn})$ and $d_{f_n}(\phi_{1mn}, \phi_{2mn}) > \eta/2$. Since $\delta_{mn} < 1/m$ the diagonal sequences $\{f_{inn}\}$, $\{u_{inn}\}$ converge to f and u_0 respectively, and $d_{f_n}(\phi_{1nn}, \phi_{2nn}) > \eta/2$. We will now prove that $d_f(\phi_{1nn}, \phi_{2nn}) > \eta/4$, which is a contradiction because $V(f) = 0$. Since the unlimited solutions ϕ_{inn} are defined on J_f for n large enough, it suffices to prove

$$\text{(6)} \qquad \begin{aligned} &\sup\{|\phi_{1nn}(t) - \phi_{2nn}(t)|_{\overline{s}} : t \in J_{f_n} - J_f\} \\ &\qquad \leq \sup\{|\phi_{1nn}(t) - \phi_{2nn}(t)|_{\overline{s}} : t \in J_f\} + \eta/4. \end{aligned}$$

To prove (6) we let θ be any positive number. If $D_f = d(u_0, U^c)$ it is clear that $D_{f_n} \leq D_f$ for every n and therefore $D_{f_n} < D_f + \theta$. If $D_f < d(u_0, U^c)$ there exists a

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point u_1 in U such that $d(u_0,u_1) < D_f + \theta/2$ and u_1 is in U_f^c. Hence there exists $\overline{u}$ in the ball $B(u_1,\theta/2)$ such that $|f(\overline{u})|_\beta > |f(u_0)|_\beta + 1$. Since $\{f_n\} \to f$ we can find n_0 large enough so that $|f_n(\overline{u})|_\beta > |f_n(u_0)|_\beta + 1$ for $n \geq n_0$, which implies $D_{f_n} \leq d(u_0,\overline{u}) < D_f + \theta$. Therefore for every $\epsilon > 0$ we have $\delta_{f_n} < \delta_f + \epsilon$ for n large enough. From (4) it is easy to prove that there exists a constant P independent of n, such that

$$|\phi_{inn}(t) - \phi_{inn}(\overline{t})|_{\overline{s}} \leq P|t-\overline{t}| \quad (t,\overline{t} \in J_{f_n}). \tag{7}$$

Assume n is large enough so that $\delta_{f_n} - \delta_f < \eta(8P)^{-1}$. Since for every $\overline{t}$ in $J_{f_n} - J_f$ there exists $t \in J_f$ such that $|\overline{t} - t| < \eta(8P)^{-1}$, it is easy to prove (6) from (7).

THEOREM 8. *Let $\mathcal{A}$ be the subset of X formed by all mappings which satisfy*

(A) *problem* (3) *has a unique solution ϕ that is defined at least on K_f;*

(B) *if $\{f_n\} \to f$ in X, $\{u_n\} \to u_0$ in $J \times X_\beta$ and* (4) *has an unlimited solution ϕ_n then for every compact set $K \subset K_f$ there exists a positive integer n_0 such that ϕ_n is defined on K for $n \geq n_0$ and $\{\phi_n\} \to \phi$ in $C(K;X_s)$.*

Then $\mathcal{A}$ is a residual subset of X.

PROOF. Let n be any positive integer. From Lemma 7 we see that $V^{-1}([0,1/n])$ contains an open set G_n which contains $V^{-1}(\{0\})$. Then the set $\mathcal{A}_k = \bigcap_{n=1}^\infty G_n$ (recall that V depends on a constant k, $0 < k < 1$) is a dense G_δ in X, because locally Lipschitz mappings are dense in X [**7**, Lemma 2]. Furthermore, if f is in $\mathcal{A}_k$ we have $V(f) = 0$. Consequently, by Lemma 5 f satisfies (A) and (B) when K_f is replaced by J_f.

Consider a sequence $\{k_n\} \to 1$. Then $\mathcal{A} = \bigcap_{n=1}^\infty \mathcal{A}_{k_n}$ is also a dense G_δ and every f in $\mathcal{A}$ satisfies (A) and (B) on K_f.

Generic existence also can be studied in the set $U \times X$. In this case we define for every pair $(u,f) \in U \times X$ the function $W: U \times X \to \mathbf{R}$ by $W(u,f) = V(f)$ for the initial point u. The arguments in Lemmas 5–7 apply equally well and we can state

THEOREM 9. *Let $\mathcal{B}$ be the subset of $U \times X$ formed by all pairs (u,f) which satisfy* (A) *and* (B) *for the mapping f and the initial condition u. Then $\mathcal{B}$ is a residual subset of $U \times X$.*

When U is separable we can apply a result of Kuratowski-Ulam [**6**] and state

THEOREM 10. *There exists a residual subset $\mathcal{A}$ of X such that for every f in $\mathcal{A}$ there exists a residual subset U' of U such that* (A) *and* (B) *are satisfied for the mapping f and every initial condition u in U'.*

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