PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 3, November 1982

ON QUASINILPOTENT SEMIGROUPS OF OPERATORS

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ABSTRACT. We construct a pair of operators such that the semigroup generated by them consists of operators which are nilpotent of index 3. The sum of the two operators, however, is not quasinilpotent.

A theorem of Levitzki [**4**] (see also [**2**]) states that any semigroup of nilpotent matrices can be put in simultaneous triangular form. This immediately implies that the sum of two elements of such a nilpotent semigroup is nilpotent.

E. Nordgren, H. Radjavi and P. Rosenthal obtained the following infinite dimensional analogue in a sequel to [**3**] that is not yet published; if A and B are operators on a Hilbert space, with A a member of a Schatten p class and B compact, and if, further, the semigroup generated by A and B consists of quasinilpotent operators, then $A+B$ is quasinilpotent. They raised the question of whether the compactness conditions are essential in this result.

In this note we construct a pair of operators such that the semigroup generated by them consists of operators which are nilpotent of index 3. The sum of the two operators, however, is not quasinilpotent.

Thue [**5**] constructed a sequence $\{\alpha_i\}$ of 0's and 1's such that no finite substring of the sequence occurs three times in a row. That is, there do not exist j and n such that $\alpha_{j+k} = \alpha_{j+n+k} = \alpha_{j+2n+k}$ for $k = 0, 1, \ldots, n-1$. Let $\{\alpha_i\}$ be such a sequence. Let A be the weighted unilateral shift with $\{\alpha_i\}$ as its weight sequence and B be the weighted unilateral shift whose weights $\{\beta_i\}$ are obtained from $\{\alpha_i\}$ by binary complementation (i.e. $\beta_i = 1-\alpha_i$). That is, there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ such that $Ae_n = \alpha_n e_{n+1}$ and $Be_n = \beta_n e_{n+1}$ for all n. Then A and B are nilpotent of index 3, since the substrings $1,1,1$, and $0,0,0$, never occur in $\{\alpha_i\}$. In a similar fashion, given an arbitrary word

$$W = A^{n_1} B^{n_2} \cdots A^{n_{m-1}} B^{n_m},$$

we can show that $W^3 e_t = 0$ for all t (and hence $W^3 = 0$). For $We_t \neq 0$ if and only if the string

$$\beta_t, \ldots, \beta_{t+(n_m-1)}, \alpha_{t+n_m}, \ldots, \alpha_{t+n_m+(n_{m-1}-1)}, \ldots, \alpha_{t+\Sigma n_i - 1}$$

consists of ones. That is,

$$\begin{aligned} \alpha_i &= 0 \quad \text{for } i = t, \ldots, t+(n_m-1) \\ &= 1 \quad \text{for } i = t+n_m, \ldots, t+n_m+(n_{m-1}-1) \\ &\ \ \vdots \\ &= 1 \quad \text{for } i = t + \textstyle\sum n_i - 1. \end{aligned}$$

Received by the editors January 18, 1982.
1980 *Mathematics Subject Classification*. Primary 47D05; Secondary 47B99.

0002-9939/82/0000-0301/$01.50

Call this substring $S_{t,W}$. If $W^3 e_t \neq 0$ then the string $S_{t,W}$, $S_{t,W}$, $S_{t,W}$, must occur in $\{\alpha_i\}$. However by the construction of $\{\alpha_i\}$ this cannot happen. Hence $W^3 e_t = 0$.

Thus the semigroup generated by A and B consists of nilpotent operators of index 3. On the other hand, $A + B$ is the unilateral shift, which has spectrum the unit disc and hence is not quasinilpotent (see [**1**]).

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