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TOTALLY REAL MINIMAL SUBMANIFOLDS IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. We give a pinching theorem with respect to the scalar curvatures of 4-dimensional conformally flat totally real minimal submanifolds in a 4-dimensional complex projective space.

1. Introduction. Among all submanifolds of an almost Hermitian manifold, there are two typical classes: one is the class of holomorphic submanifolds and the other is the class of totally real submanifolds. There have been many results in the theory of holomorphic submanifolds; on the other hand, there have been only a few results in the theory of totally real submanifolds.

H. Naitoh [**2**], M. Takeuchi [**3**] classified submanifolds in a real and complex space form with parallel second fundamental form.

Among such examples, there exist three n-dimensional conformally flat totally real minimal submanifolds in a complex projective space P_n of constant holomorphic sectional curvature 4:

(i) a totally geodesic submanifold,

(ii) a flat torus,

(iii) a Riemannian product of $S^1(\sin a \cos a)$ and $S^{n-1}(\sin a)$, where $S^n(r)$ is an n-dimensional sphere with radius r and $\tan a = \sqrt{n}$.

The purpose of this paper is to give a characterization of (ii) and (iii) of 4 dimension.

THEOREM. *Let M be a 4-dimensional compact orientable conformally flat totally real minimal submanifold in P_4. If M has nonnegative Euler number and the scalar curvature ρ of M is between* 0 *and* 15/2, *then ρ is* 0 *or* 15/2 *and M is* (ii) ($\rho = 0$), (iii) ($\rho = 15/2$) *or its covering spaces.*

REMARK. If $n = 4$, B. Y. Chen and K. Ogiue's result [**1**] implies that every compact totally real minimal submanifold in P_4 with $\rho \geqslant 64/7$ is totally geodesic ($\rho = 12$).

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2. Proof of Theorem. We use the same notations and terminologies as in [**1**]. It was proved in [**1**] that the second fundamental form of the immersion satisfies

$$(1)\quad \tfrac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 + \sum_{i,j}\operatorname{tr}\left(A_{i^*}A_{j^*} - A_{j^*}A_{i^*}\right)^2 - \sum_{i,j}\left(\operatorname{tr} A_{i^*}A_{j^*}\right)^2 + 5\|\sigma\|^2.$$

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Since

$$\sum_{i,j} \operatorname{tr}(A_{i^*}A_{j^*} - A_{j^*}A_{i^*})^2 = -\sum_{i,j,k,l}\left(\sum_m \left(h^{i^*}_{km}h^{j^*}_{lm} - h^{j^*}_{km}h^{i^*}_{lm}\right)\right)^2,$$

this, together with the equation of Gauss, implies

$$\sum_{i,j} \operatorname{tr}(A_{i^*}A_{j^*} - A_{j^*}A_{i^*})^2 = \|R\|^2 + 4\rho - 24. \tag{2}$$

By the same argument as above, we have

$$\sum_{i,j} (\operatorname{tr} A_{i^*}A_{j^*})^2 = \|S\|^2 - 6\rho + 36. \tag{3}$$

Combining (1) with (2) and (3), we obtain

$$\tfrac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 - \|R\|^2 - \|S\|^2 + 5\rho. \tag{4}$$

From the assumption that M is conformally flat, we obtain

$$\|R\|^2 - 2\|S\|^2 + \tfrac{1}{3}\rho^2 = 0, \tag{5}$$

which, together with (4), asserts

$$\tfrac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 - 3\|S\|^2 + \tfrac{1}{3}\rho^2 + 5\rho.$$

Taking the integrals of the both sides of it and using Green's theorem, we have

$$\int_M \|\nabla'\sigma\|^2 * l_M = \int_M \{3\|S\|^2 - \tfrac{1}{3}\rho^2 - 5\rho\} * l_M. \tag{6}$$

On the other hand, by the Gauss-Bonnet theorem, the Euler number $\chi(M)$ of M is given by

$$\chi(M) = \frac{1}{32\pi^2}\int_M \{\|R\|^2 - 4\|S\|^2 + \rho^2\} * l_M.$$

It follows from (5) that

$$\chi(M) = \frac{1}{32\pi^2}\int_M \left\{\frac{2}{3}\rho^2 - 2\|S\|^2\right\} * l_M. \tag{7}$$

Combining (6) with (7), we get an integral formula:

$$48\pi^2\chi(M) + \int_M \|\nabla'\sigma\|^2 * l_M = \int_M \frac{2}{3}\rho\left\{\rho - \frac{15}{2}\right\} * l_M.$$

Theorem follows from the integral formula and results in [**2**, **3**].

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