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PRODUCTS OF CW-COMPLEXES

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ABSTRACT. We show that Liu's characterization for the product $K \times L$ to be a CW-complex is independent of the usual axioms of set theory.

1. Introduction. The concept of CW-complex due to J. H. C. Whitehead [**8**] is well known. Recall that a space K is a *CW-complex*, if it is a complex with cells $\{e_\alpha\}$ such that each e_α is contained in a finite subcomplex, and K has the weak topology with respect to a closed cover $\{\bar{e}_\alpha\}$; that is, $F \subset K$ is closed in K if $F \cap \bar{e}_\alpha$ is closed for each $\bar{e}_\alpha$. Every Whitehead complex introduced by C. H. Dowker [**1**] is a CW-complex with cells $\{e_\alpha\}$ such that each $\bar{e}_\alpha$ is a subcomplex.

Let K be a CW-complex with cells $\{e_\alpha\}$. Then K is called *locally finite*; *locally countable*, if for each $x \in K$, there is respectively a finite; countable subcomplex A of K with $x \in \operatorname{int} A$. Hence K is locally finite; locally countable, if and only if a closed cover $\{\bar{e}_\alpha\}$ of the space K is so respectively.

Liu Ying-ming [**3**], assuming the continuum hypothesis (CH), gave the following necessary and sufficient condition for the product of two CW-complexes to be a CW-complex.

(CH). Let K and L be CW-complexes. Then $K \times L$ is a CW-complex if and only if K or L is locally finite, or K and L are locally countable.

On the other hand, assuming (CH), we gave a necessary and sufficient condition for the product of two closed images of metric spaces to be a k-space [**5**]. G. Gruenhage [**2**] showed that this characterization is equivalent to a certain set-theoretic axiom weaker than (CH).

In this paper, analogously we shall show that Liu's result is in fact equivalent to this set-theoretic axiom. And also, if $K = L$, this result is valid without any set-theory beyond ZFC. These are affirmative answers to the questions in [**6**]. Many of the results in this paper were also obtained by Zhou Hao-xuan in his paper [**9**]. The author wishes to thank him for his translation of Liu's paper [**3**].

2. Results. First of all, we shall recall the well-known examples below. Let S_ω be a *sequential fan*; that is, it is the quotient space obtained from the topological sum of ω convergent sequences by identifying all the limit points. S_α, $\alpha > \omega$, are similarly defined replacing "ω" by "α". We also need another canonical example S_2. That is, S_2 is the space $(N \times N) \cup N \cup \{0\}$ with each point of $N \times N$ isolated, N is the set

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of natural numbers. A basis of neighborhoods of $n \in N$ consists of all sets of the form $\{n\} \cup \{(m, n): m \geqslant m_0\}$. And U is a neighborhood of 0 if and only if $0 \in U$ and U is a neighborhood of all but finitely many $n \in N$. We remark that S_ω is the perfect image of S_2 by identifying all points of a compact subst $N \cup \{0\}$ of S_2. This will be used later.

In [7], we showed that the metrizability of certain quotient images of metric spaces can be characterized by whether or not they contain copies of S_ω and S_2. As for CW-complexes, we have the following by invoking [**6**, Proposition 2.3] and the well-known fact that every CW-complex is locally finite if and only if it is metrizable.

LEMMA 1. *Let K be a* CW*-complex. Then the following are equivalent.*

(1) *K is metrizable.*

(2) *K is locally finite.*

(3) *K contains no closed copy of S_ω and no S_2.*

Now, we consider the products of CW-complexes in terms of a certain set-theoretic axiom weaker than (CH).

Let ${}^N N$ be the set of all functions from N into N. For $f, g \in {}^N N$, we define $f \leqslant g$ if and only if $\{n \in N; f(n) > g(n)\}$ is finite. For infinite cardinal α, by $BF(\alpha)$ we mean the following assertion:

$BF(\alpha)$: If $F \subset {}^N N$ has cardinality less than α, then there exists $g \in {}^N N$ such that $f \leqslant g$ for all $f \in F$.

It is well known that Martin's Axiom implies that $BF(\alpha)$ holds for all α less than or equal to the continuum. It is easy to show that (CH) implies that $BF(\omega_2)$ is false. As for $BF(\alpha)$, G. Gruenhage [2] gave the following equivalence in terms of products of k-spaces, α^+ denotes the least cardinal greater than α. Recall that a space X is a *k-space* if it has the weak topology with respect to the cover consisting of all compact subsets of X.

(*) $BF(\alpha^+)$ holds if and only if $S_\omega \times S_\alpha$ is a k-space.

To apply this result (*) to the products of CW-complexes, let I_α be the quotient space obtained from the topological sum of α closed unit intervals [0, 1] by identifying all the zero points. Then the result (*) suggests the following.

LEMMA 2. *$BF(\alpha^+)$ holds if and only if $I_\omega \times I_\alpha$ is a* CW*-complex.*

PROOF. If $BF(\alpha^+)$ holds, then it follows from the proof of [**2**, Lemma 1] that $I_\omega \times I_\alpha$ is sequential, hence a CW-complex by [**6**, Proposition 2.5]. If $I_\omega \times I_\alpha$ is a CW-complex, hence a k-space, then $S_\omega \times S_\alpha$ is a k-space, for $S_\omega \times S_\alpha$ is closed in $I_\omega \times I_\alpha$. Hence, by the result (*), $BF(\alpha^+)$ holds.

LEMMA 3. *Suppose that e is a cell of a* CW*-complex K such that $x \in e$ and each neighborhood of x in e meets at least ω_1 many boundaries of cells of K. Suppose that L is a* CW*-complex which is not locally finite. Then $K \times L$ is not a* CW*-complex if $BF(\omega_2)$ is false.*

PROOF. Since e is first countable, there is a decreasing local base $\{G_n;\ n \in N\}$ of x in e. By the hypothesis, there exist pairwise disjoint collections Ω_n, $n \in N$, of cells of K such that $|\Omega_n| = \omega_1$, and boundary of each cell of Ω_n meets G_n. For each $e \in \Omega_n$, let $x(e) \in \partial e \cap G_n$, and let $\{S_n(e);\ n \in N\}$ be a decreasing local base of $x(e)$ in $\bar{e}$. Then there exists a subset $\{x(e, n);\ n \in N\}$ of e with $x(e, n) \in S_n(e)$. Now, since the CW-complex L is not locally finite, by Lemma 1, L contains a closed copy of S_ω or S_2. Suppose that $K \times L$ is a CW-complex, hence a k-space. If L contains a closed copy of S_ω, then $K \times S_\omega$ is a k-space. If L contains a closed copy S_2, then $K \times S_2$ is a k-space. But, S_ω is the perfect image of S_2, so $K \times S_\omega$ is the perfect image of $K \times S_2$. Thus $K \times S_\omega$ is a k-space. Hence, in any case, $K \times S_\omega$ is a k-space.

Now, since $BF(\omega_2)$ is false, there is a collection $\{f_\alpha;\ f_\alpha\colon N \to N,\ \alpha < \omega_1\}$ such that if $f\colon N \to N$, then there exists $\alpha < \omega_1$ with $f_\alpha(n) > f(n)$ for infinitely many $n \in N$. Since each Ω_n has cardinality of ω_1, we can put $\Omega_n = \{e^n_\alpha;\ \alpha < \omega_1\}$. For each $j \in N$, let

$$H_j = \bigcup_{\alpha \in \omega_1} \{(x(e^j_\alpha, n), (n, m));\ f_\alpha(n) \geqslant m\},$$

and $H = \bigcup_{j \in N} H_j$, where (n, m) is the mth term of the nth sequence in S_ω. Then it is easy to show that $H \cap C$ is finite for each compact subset C of $K \times S_\omega$, because each compact subset of $K(S_\omega)$ meets only finitely many cells of K (sequences in S_ω). Since $K \times S_\omega$ is a k-space, H is closed in $K \times S_\omega$. We obtain a contradiction by showing that $(x, \infty) \in \overline{H}$, where ∞ is the nonisolated point in S_ω. This contradiction implies that $K \times L$ is not a CW-complex. Thus it remains to show that $(x, \infty) \in \overline{H}$. Let $W = U \times V$ be a neighborhood of (x, ∞) in $K \times S_\omega$, and let $G_l \subset U$. For each $n \in N$, since V is a neighborhood of ∞ in S_ω, there exists $n' \in N$ such that $n' > n$ and $(n, m) \in V$ if $m \geqslant n'$. Let $f\colon N \to N$ be defined by $f(n) = n'$. Then there exists $\alpha_0 < \omega_1$ such that $f_{\alpha_0}(n) > f(n)$ for infinitely many $n \in N$. Since $x(e^l_{\alpha_0}, n) \to x(e^l_{\alpha_0}) \in U$, and U is open in K, there exists $n_0 \in N$ with $x(e^l_{\alpha_0}, n_0) \in U$ and $f_{\alpha_0}(n_0) > f(n_0)$. Then, $(x(e^l_{\alpha_0}, n_0), (n_0, f_{\alpha_0}(n_0))) \in H_l \cap (U \times V)$. This implies that $(x, \infty) \in \overline{H}$.

LEMMA 4 [3, LEMMA 3]. *Let K be a* CW*-complex. Suppose that for $x \in K$ and a cell e of K with $x \in e$, there is a neighborhood U of x in e such that $|\{e \in K;\ \partial e \cap U \neq \varnothing\}| \leqslant \omega$. Then K is locally countable at x.*

Now we are ready for the main result concerning products.

THEOREM 5. *Let K and L be* CW*-complexes. Then the following are equivalent.*

(1) *$BF(\omega_2)$ is false.*

(2) *$K \times L$ is a* CW*-complex if and only if K or L is locally finite, otherwise K and L are locally countable.*

PROOF. (2) $\to$ (1): Suppose that $BF(\omega_2)$ holds. Then, by Lemma 2, $I_\omega \times I_{\omega_1}$ is a CW-complex. But, in this case, the "only if" part does not hold. Hence $BF(\omega_2)$ is false.

(1) → (2): The "if" part of (2) is well known. Indeed, it essentially follows from results of [**8**, (H)] and [**4**, Lemma 2.1]. So we prove the "only if" part. Suppose that K is not locally countable and also L is not locally finite. Since K is not locally countable, by Lemma 4 there is $x \in e$ such that every neighborhood x in e meets at least ω_1 many boundaries of cells of K. Thus, since $BF(\omega_2)$ is false, $K \times L$ is not a CW-complex by Lemma 3. This is a contradiction. Hence L is locally finite if K is not locally countable. Similarly, K is locally finite if L is not locally countable. That completes the proof.

LEMMA 6. *Suppose that e; τ is respectively a cell of a* CW-*complex K; L such that $x \in e$; $y \in \tau$, and each neighborhood of x in e; y in τ meets at least ω_1 many boundaries of cells of K; L. Then $K \times L$ is not a* CW-*complex.*

PROOF. For each $\alpha < \omega_1$, let f_α: $\omega_1 \to N$ be a function such that f_α restricted to α is a one-to-one map onto N. Let $\{G_n;\ n \in N\}$, and $\{x(e_\alpha^j, n);\ j, n \in N, \alpha < \omega_1\}$ be the same as in the proof of Lemma 3. Similarly define $\{G_n':\ n \in N\}$ and $\{y(\tau_\alpha^j, n);\ j, n \in N,\ \alpha < \omega_1\}$ in L. For each $j \in N$, let $M_j = \bigcup_{\alpha,\beta<\omega_1}\{(x(e_\alpha^j, n), y(\tau_\beta^j, f_\beta(\alpha)));\ n < f_\beta(\alpha)\}$, and $M = \bigcup_{j \in N} M_j$. Let us now suppose that $K \times L$ is a CW-complex, hence a k-space. Then M is closed in $K \times L$, because $M \cap C$ is finite for each compact subset C of $K \times L$. However, we have a contradiction that $(x, y) \in \overline{M} - M$ by referring to the proof of [**2**, Lemma 5], hence $K \times L$ is not a CW-complex. Indeed, to show $(x, y) \in \overline{M}$, let $U \times V$ be a neighborhood of (x, y) in $K \times L$, and $G_l \subset U$, $G_l' \subset V$. Then there is a function g: $\omega_1 \to N$ such that for each $\alpha < \omega_1$, $\{x(e_\alpha^l, n);\ n \geqslant g(\alpha)\} \subset U$ and $\{y(\tau_\alpha^l, n);\ n \geqslant g(\alpha)\} \subset V$. Thus there is $n_0 \in N$ and an uncountable subset A of ω_1 with $g(\alpha) = n_0$ if $\alpha \in A$. Let γ be an element of A which has infinitely many predecessors in A. Hence there is $\delta \in A$ with $\delta < \gamma$ and $f_\gamma(\delta) = m > n_0$. Thus $(x(e_\delta^l, n_0), y(\tau_\gamma^l, f_\gamma(\delta))) \in M \cap (U \times V)$. Hence $M \cap (U \times V) \neq \varnothing$, which implies $(x, y) \in \overline{M}$.

By the previous lemma and Lemma 4, we have

THEOREM 7. *If $K \times L$ is a* CW-*complex, then K or L is locally countable. When $K = L$, the converse holds.*

REMARK. Let K be a CW-complex. For each infinite cardinal α, let us call K *locally* α, if $\alpha = \min\{\gamma$, for any $x \in K$, there is a subcomplex A consisting of $\leqslant \gamma$ many cells with $x \in \operatorname{int} A\}$ (hence, $\alpha = \min\{\gamma$; for any $x \in K$, there is a neighborhood of x which meets $\leqslant \gamma$ many closed cells $\bar{e}\}$).

Then we have the following analogue to Theorem 5 by Theorem 7, Lemma 2, and slight modifications of Lemmas 3 and 4 (cf. [**9**, Theorem 2.7]).

THEOREM. *$BF(\alpha^+)$ is false if and only if whenever $K \times L$ is* CW-*complex, K or L is locally finite, otherwise one of K, L is locally countable and another is locally $< \alpha$.*

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