MORE ON M. E. RUDIN'S DOWKER SPACE

KLAAS PIETER HART

Abstract. It is shown that M. E. Rudin’s Dowker space is finitely-fully normal and orthocompact, thus answering questions of Mansfield and Scott.

0. Introduction. In [Ma] Mansfield defined the notions of $\kappa$-full normality and finite-full normality. One of the questions he raised was, whether there exists a finitely-fully normal space which is not an $\omega_0$-fully normal space.

In [Sc] Scott asked whether M. E. Rudin’s Dowker space [Ru] is orthocompact. We answer both questions simultaneously by showing that the above-mentioned space is both finitely-fully normal and orthocompact. Mansfield’s question is hereby answered since in [Ma] he showed that almost $\omega_0$-fully normal spaces are countably paracompact. Almost $\kappa$-full normality will not be defined here; it suffices to know that it is weaker than $\kappa$-full normality.

1. Definitions and preliminaries.

1.0 $\kappa$-full normality and orthocompactness. Let $Y$ be a topological space, $\mathcal{U}$ an open cover of $Y$ and $\kappa \geq 2$ a cardinal. An open cover $\mathcal{V}$ is said to be a $\kappa$-star (finite-star) refinement of $\mathcal{U}$ if for all $\mathcal{V}' \subseteq \mathcal{V}$ with $|\mathcal{V}'| \leq \kappa$ ($\mathcal{V}'$ finite) and $\bigcap \mathcal{V}' \neq \emptyset$ there is a $U \in \mathcal{U}$ with $\bigcup \mathcal{V}' \subseteq U$, and $\mathcal{V}$ is a $Q$-refinement of $\mathcal{U}$ if $\mathcal{V}$ refines $\mathcal{U}$ and $\bigcap \mathcal{V}'$ is open for all $\mathcal{V}' \subseteq \mathcal{V}$. (Recent practice is to call $Q$-refinements interior-preserving open refinements.)

$Y$ is called $\kappa$-fully (finitely-fully) normal [Ma] if every open cover of $Y$ has a $\kappa$-star (finite-star) refinement. $Y$ is called orthocompact [Se] if every open cover of $Y$ has a $Q$-refinement.

1.1 M. E. Rudin’s Dowker space. Let $F = \prod_{n=1}^\infty (\omega_n + 1)$ endowed with the box topology. Furthermore let $X' = \{ f \in F : \forall n \in \mathbb{N} \text{ cf}(f(n)) > \omega_0 \}$ and $X = \{ f \in X' : \exists i \in \mathbb{N} : \forall n \in \mathbb{N} \text{ cf}(f(n)) < \omega_i \}$. Then $X$ is M. E. Rudin’s Dowker space [Ru].

We give an alternative description of the canonical base for $X'$ (and $X$). For $f, g \in F$ we say

\[ f < g \text{ if } f(n) < g(n) \text{ for all } n, \]
\[ f \leq g \text{ if } f(n) \leq g(n) \text{ for all } n. \]

For $f, g \in F$ with $f < g$ we let

\[ U_{f,g} = \{ h \in X' : f < h \leq g \} \]
and
\[ U_{f,g} = U_{f,g} \cap X. \]
Then
\[ \{ U_{f,g} : f, g \in F, f < g \} \]
is a base for the topology of \( X^\ell \). Notice that the basic open sets are convex in the partial order \( \leq \) on \( X \), a fact we will use in the proof of Theorem 2.2.

2. The main result. In this section we prove using the results from [Ru] and [Ha] that the Dowker space \( X \) is finitely-fully normal and orthocompact. First we formulate a lemma, the proof of which can be found (implicitly) in the proof in [Ru] that \( X \) is collectionwise normal.

2.0 Lemma. a. Every open cover of \( X' \) has a disjoint refinement consisting of basic open sets.

b. If \( A, B \subseteq X \) are closed and disjoint then
\[ \text{Cl}_{X'} A \cap \text{Cl}_{X'} B = \emptyset. \]

The next result is from [Ha].

2.1 Lemma. For all \( n \in \mathbb{N} \): \( (X')^n \) is homeomorphic to \( X' \), and the homeomorphism can be chosen to map \( X^n \) onto \( X \).

Now we are ready to prove the main result.

2.2 Theorem. The space \( X \) is both 2-fully normal and orthocompact.

Proof. Let \( \mathcal{U} \) be a basic open cover of \( X \). Put \( U = \bigcup\{ 0 \times 0 \times 0 : 0 \in \mathcal{U} \} \); \( U \) is a neighborhood of \( \{(x, x, x) : x \in X\} \) in \( X^3 \). Using 2.1 and 2.0b find a neighborhood \( U' \) of \( \{(x, x, x) : x \in X'\} \) in \( (X')^3 \) such that \( U' \cap X^3 = U \).

For \( x \in X' \setminus X \), choose \( U_x \subseteq x \) open such that \( U_x \cap X^3 = U \).

By 2.0a let \( \mathcal{O}' \) be a disjoint basic open refinement of the open cover
\[ \{ X' \setminus \text{Cl}_{X'}(X \setminus 0) \}_{0 \in \mathcal{U}} \cup \{ U_x \}_{x \in X' \setminus X}. \]

Let \( \emptyset = \{ 0' \cap X : 0' \in \mathcal{O}' \} \).

Let 0 \in \emptyset and \( \{ x, y, z \} \subseteq 0 \).

Then \( \{ x, y, z \} \subseteq \text{some} V \in \mathcal{U} \) or \( \{ x, y, z \} \subseteq \text{some} U_p \), but then \( \{ x, y, z \} \subseteq U_p \cap X^3 \subseteq U \), so \( \{ x, y, z \} \subseteq V^3 \) for some \( V \in \mathcal{U} \) in any case. This implies that \( \{ x, y, z \} \subseteq V \).

For each \( 0 \in \emptyset \) define \( \mathcal{W}_0 \) as follows: \( 0 = U_{p,q} \) for some \( p, q \in F \), so put \( \mathcal{W}_0 = \{ U_{p,q} : x \in 0 \} \). Let \( \mathcal{W} = \bigcup \{ \mathcal{W}_0 : 0 \in \emptyset \} \). Then \( \mathcal{W} \) is both a 2-star and a \( Q \)-refinement of \( \mathcal{U} \).

First, assume \( U_{p,x} \cap U_{q,y} \neq \emptyset \) for some \( U_{p,x} \) and \( U_{q,y} \) in \( \mathcal{W} \). Then \( x \) and \( y \) are elements of the same \( 0 \in \emptyset \) and hence \( p = q \). Define \( p' \) by \( p'(n) = p(n) + \omega_1 \) \((n \in \mathbb{N})\); then \( p < p' \leq x \) and \( p' \in X \), so \( p' \in 0 \).

Pick \( u \in \mathcal{U} \) such that \( \{ p', x, y \} \subseteq U \). Since \( U \) is basic (and hence \( \leq \)-convex) and \( U_{p,z} = \{ t : p' \leq t \leq z \} \) for \( z = x, y \), it follows that \( U_{p,x} \cup U_{p,y} \subseteq U \). So \( \mathcal{W} \) is a 2-star refinement of \( \mathcal{U} \). Second, let \( \mathcal{W}' \subseteq \mathcal{W} \) with \( \cap \mathcal{W}' = \emptyset \). Then all \( W \in \mathcal{W}' \) are
510
K. P. HART

contained in the same \(0 \in \mathcal{O}\), so \(\mathcal{W}' = \{U_{p,x} : x \in A\}\) for some subset \(A\) of \(0\), where \(0 = U_{p,q}\). Define \(f\) by \(f(n) = \min\{x(n) : x \in A\}\). Then \(\cap \mathcal{W}' = U_{p,f}\) is open. So \(\mathcal{W}\) is a \(Q\)-refinement of \(\mathcal{U}\). \(\square\)

It now follows easily that \(X\) is finitely-fully normal:

2.3 Corollary. \(X\) is finitely-fully normal.

Proof. Let \(\mathcal{U}\) be an open cover of \(X\). Let \(\mathcal{V}_1\) be a 2-star refinement of \(\mathcal{U}\), and (inductively) let \(\mathcal{V}_{n+1}\) be a 2-star refinement of \(\mathcal{V}_n\) (\(n \in \mathbb{N}\)). Since \(X\) is a \(P\)-space (\(G_\delta\)'s are open) we can take the common refinement of all \(\mathcal{V}_n\); call it \(\mathcal{V}\). Let \(\mathcal{V}' \subseteq \mathcal{V}\) be finite with \(\cap \mathcal{V}' \neq \emptyset\). Pick \(n \in \mathbb{N}\) such that \(2^n \geq |\mathcal{V}'|\). Since \(\mathcal{V}\) refines \(\mathcal{V}_n\) and since \(\mathcal{V}_n\) is a \(2^n\)-star refinement of \(\mathcal{U}\), it follows that \(\bigcup \mathcal{V}'\) is contained in some \(U \in \mathcal{U}\). \(\square\)

References

[Ru] M. E. Rudin, A normal space \(X\) for which \(X \times I\) is not normal, Fund. Math. 73 (1971), 179–186.

SUBFACULTEIT WISKUNDE, VRIJE UNIVERSITEIT, DE BOELELAAN 1081, 1081 HV AMSTERDAM, THE NETHERLANDS