

MORE ON M. E. RUDIN'S DOWKER SPACE

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ABSTRACT. It is shown that M. E. Rudin's Dowker space is finitely-fully normal and orthocompact, thus answering questions of Mansfield and Scott.

0. Introduction. In [Ma] Mansfield defined the notions of κ -full normality and finite-full normality. One of the questions he raised was, whether there exists a finitely-fully normal space which is not an ω_0 -fully normal space.

In [Sc] Scott asked whether M. E. Rudin's Dowker space [Ru] is orthocompact. We answer both questions simultaneously by showing that the above-mentioned space is both finitely-fully normal and orthocompact. Mansfield's question is hereby answered since in [Ma] he showed that almost ω_0 -fully normal spaces are countably paracompact. Almost κ -full normality will not be defined here; it suffices to know that it is weaker than κ -full normality.

1. Definitions and preliminaries.

1.0 κ -full normality and orthocompactness. Let Y be a topological space, \mathcal{Q} an open cover of Y and $\kappa \geq 2$ a cardinal. An open cover \mathcal{V} is said to be a κ -star (finite-star) refinement of \mathcal{Q} if for all $\mathcal{V}' \subseteq \mathcal{V}$ with $|\mathcal{V}'| \leq \kappa$ (\mathcal{V}' finite) and $\bigcap \mathcal{V}' \neq \emptyset$ there is a $U \in \mathcal{Q}$ with $\bigcup \mathcal{V}' \subseteq U$, and \mathcal{V} is a Q -refinement of \mathcal{Q} if \mathcal{V} refines \mathcal{Q} and $\bigcap \mathcal{V}'$ is open for all $\mathcal{V}' \subseteq \mathcal{V}$. (Recent practice is to call Q -refinements interior-preserving open refinements.)

Y is called κ -fully (finitely-fully) normal [Ma] if every open cover of Y has a κ -star (finite-star) refinement. Y is called orthocompact [Sc] if every open cover of Y has a Q -refinement.

1.1 M. E. Rudin's Dowker space. Let $F = \prod_{n=1}^{\infty} (\omega_n + 1)$ endowed with the box topology. Furthermore let $X' = \{f \in F: \forall n \in \mathbf{N} \text{ cf}(f(n)) > \omega_0\}$ and $X = \{f \in X': \exists i \in \mathbf{N}: \forall n \in \mathbf{N} \text{ cf}(f(n)) < \omega_i\}$. Then X is M. E. Rudin's Dowker space [Ru].

We give an alternative description of the canonical base for X' (and X). For $f, g \in F$ we say

$$\begin{aligned} f < g & \text{ if } f(n) < g(n) \text{ for all } n, \\ f \leq g & \text{ if } f(n) \leq g(n) \text{ for all } n. \end{aligned}$$

For $f, g \in F$ with $f < g$ we let

$$U'_{f,g} = \{h \in X': f < h \leq g\}$$

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and

$$U_{f,g} = U'_{f,g} \cap X.$$

Then

$$\{U_{f,g}^{(\cdot)} : f, g \in F, f < g\}$$

is a base for the topology of $X^{(\cdot)}$. Notice that the basic open sets are convex in the partial order \leq on X , a fact we will use in the proof of Theorem 2.2.

2. The main result. In this section we prove using the results from [Ru] and [Ha] that the Dowker space X is finitely-fully normal and orthocompact. First we formulate a lemma, the proof of which can be found (implicitly) in the proof in [Ru] that X is collectionwise normal.

2.0 LEMMA. a. Every open cover of X' has a disjoint refinement consisting of basic open sets.

b. If $A, B \subseteq X$ are closed and disjoint then

$$Cl_{X'}A \cap Cl_{X'}B = \emptyset. \quad \square$$

The next result is from [Ha].

2.1 LEMMA. For all $n \in \mathbb{N}$: $(X')^n$ is homeomorphic to X' , and the homeomorphism can be chosen to map X^n onto X .

Now we are ready to prove the main result.

2.2 THEOREM. The space X is both 2-fully normal and orthocompact.

PROOF. Let \mathcal{U} be a basic open cover of X . Put $U = \bigcup \{0 \times 0 \times 0 : 0 \in \mathcal{U}\}$; U is a neighborhood of $\{\langle x, x, x \rangle : x \in X\}$ in X^3 . Using 2.1 and 2.0b find a neighborhood U' of $\{\langle x, x, x \rangle : x \in X'\}$ in $(X')^3$ such that $U' \cap X^3 = U$.

For $x \in X' \setminus X$, choose $U_x \ni x$ open such that $U_x^3 \subseteq U'$.

By 2.0a let \mathcal{O}' be a disjoint basic open refinement of the open cover

$$\{X' \setminus Cl_{X'}(X \setminus 0)\}_{0 \in \mathcal{U}} \cup \{U_x\}_{x \in X' \setminus X}.$$

Let $\mathcal{O} = \{O' \cap X : O' \in \mathcal{O}'\}$.

Let $0 \in \mathcal{O}$ and $\{x, y, z\} \subseteq 0$.

Then $\{x, y, z\} \subseteq$ some $V \in \mathcal{U}$ or $\{x, y, z\} \subseteq$ some U_p , but then $\langle x, y, z \rangle \in U_p^3 \cap X^3 \subseteq U$, so $\langle x, y, z \rangle \in V^3$ for some $V \in \mathcal{U}$ in any case. This implies that $\{x, y, z\} \subseteq V$.

For each $0 \in \mathcal{O}$ define \mathcal{W}_0 as follows: $0 = U_{p,q}$ for some $p, q \in F$, so put $\mathcal{W}_0 = \{U_{p,x} : x \in 0\}$. Let $\mathcal{W} = \bigcup \{\mathcal{W}_0 : 0 \in \mathcal{O}\}$. Then \mathcal{W} is both a 2-star and a \mathcal{Q} -refinement of \mathcal{U} .

First, assume $U_{p,x} \cap U_{q,y} \neq \emptyset$ for some $U_{p,x}$ and $U_{q,y}$ in \mathcal{W} . Then x and y are elements of the same $0 \in \mathcal{O}$ and hence $p = q$. Define p' by $p'(n) = p(n) + \omega_1$ ($n \in \mathbb{N}$); then $p < p' \leq x, y$ and $p' \in X$, so $p' \in 0$.

Pick $u \in \mathcal{U}$ such that $\{p', x, y\} \subseteq U$. Since U is basic (and hence \leq -convex) and $U_{p,z} = \{t : p' \leq t \leq z\}$ for $z = x, y$, it follows that $U_{p,x} \cup U_{p,y} \subseteq U$. So \mathcal{W} is a 2-star refinement of \mathcal{U} . Second, let $\mathcal{W}' \subseteq \mathcal{W}$ with $\bigcap \mathcal{W}' \neq \emptyset$. Then all $W \in \mathcal{W}'$ are

contained in the same $0 \in \mathcal{O}$, so $\mathcal{W}' = \{U_{p,x} : x \in A\}$ for some subset A of 0 , where $0 = U_{p,q}$. Define f by $f(n) = \min\{x(n) : x \in A\}$. Then $\bigcap \mathcal{W}' = U_{p,f}$ is open. So \mathcal{W} is a \mathcal{Q} -refinement of \mathcal{U} . \square

It now follows easily that X is finitely-fully normal:

2.3 COROLLARY. X is finitely-fully normal.

PROOF. Let \mathcal{U} be an open cover of X . Let \mathcal{V}_1 be a 2-star refinement of \mathcal{U} , and (inductively) let \mathcal{V}_{n+1} be a 2-star refinement of \mathcal{V}_n ($n \in \mathbf{N}$). Since X is a P -space (G_δ 's are open) we can take the common refinement of all \mathcal{V}_n ; call it \mathcal{V} . Let $\mathcal{V}' \subseteq \mathcal{V}$ be finite with $\bigcap \mathcal{V}' \neq \emptyset$. Pick $n \in \mathbf{N}$ such that $2^n \geq |\mathcal{V}'|$. Since \mathcal{V} refines \mathcal{V}_n and since \mathcal{V}_n is a 2^n -star refinement of \mathcal{U} , it follows that $\bigcup \mathcal{V}'$ is contained in some $U \in \mathcal{U}$. \square

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