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MORE ON M. E. RUDIN'S DOWKER SPACE

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ABSTRACT. It is shown that M. E. Rudin's Dowker space is finitely-fully normal and orthocompact, thus answering questions of Mansfield and Scott.

0. Introduction. In **[Ma]** Mansfield defined the notions of κ-full normality and finite-full normality. One of the questions he raised was, whether there exists a finitely-fully normal space which is not an ω_0-fully normal space.

In **[Sc]** Scott asked whether M. E. Rudin's Dowker space **[Ru]** is orthocompact. We answer both questions simultaneously by showing that the above-mentioned space is both finitely-fully normal and orthocompact. Mansfield's question is hereby answered since in **[Ma]** he showed that almost ω_0-fully normal spaces are countably paracompact. Almost κ-full normality will not be defined here; it suffices to know that it is weaker than κ-full normality.

1. Definitions and preliminaries.

1.0 *κ-full normality and orthocompactness.* Let Y be a topological space, $\mathcal{U}$ an open cover of Y and $\kappa \geqslant 2$ a cardinal. An open cover $\mathcal{V}$ is said to be a κ-star (finite-star) refinement of $\mathcal{U}$ if for all $\mathcal{V}' \subseteq \mathcal{V}$ with $|\mathcal{V}'| \leqslant \kappa$ ($\mathcal{V}'$ finite) and $\bigcap \mathcal{V}' \neq \varnothing$ there is a $U \in \mathcal{U}$ with $\bigcup \mathcal{V}' \subseteq U$, and $\mathcal{V}$ is a Q-refinement of $\mathcal{U}$ if $\mathcal{V}$ refines $\mathcal{U}$ and $\bigcap \mathcal{V}'$ is open for all $\mathcal{V}' \subseteq \mathcal{V}$. (Recent practice is to call Q-refinements interior–preserving open refinements.)

Y is called κ-fully (finitely-fully) normal **[Ma]** if every open cover of Y has a κ-star (finite-star) refinement. Y is called orthocompact **[Sc]** if every open cover of Y has a Q-refinement.

1.1 *M. E. Rudin's Dowker space.* Let $F = \prod_{n=1}^{\infty}(\omega_n + 1)$ endowed with the box topology. Furthermore let $X' = \{f \in F\colon \forall n \in \mathbf{N}\ \mathrm{cf}(f(n)) > \omega_0\}$ and $X = \{f \in X'\colon \exists i \in \mathbf{N}\colon \forall n \in \mathbf{N}\ \mathrm{cf}(f(n)) < \omega_i\}$. Then X is M. E. Rudin's Dowker space **[Ru]**.

We give an alternative description of the canonical base for X' (and X). For f, $g \in F$ we say

$$f < g \text{ if } f(n) < g(n) \text{ for all } n,$$
$$f \leqslant g \text{ if } f(n) \leqslant g(n) \text{ for all } n.$$

For $f, g \in F$ with $f < g$ we let

$$U'_{f,g} = \{h \in X'\colon f < h \leqslant g\}$$

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and

$$U_{f,g} = U'_{f,g} \cap X.$$

Then

$$\{U^{(\prime)}_{f,g}: f, g \in F, f < g\}$$

is a base for the topology of $X^{(\prime)}$. Notice that the basic open sets are convex in the partial order $\leqslant$ on X, a fact we will use in the proof of Theorem 2.2.

2. The main result. In this section we prove using the results from **[Ru]** and **[Ha]** that the Dowker space X is finitely-fully normal and orthocompact. First we formulate a lemma, the proof of which can be found (implicitly) in the proof in **[Ru]** that X is collectionwise normal.

2.0 LEMMA. a. *Every open cover of X' has a disjoint refinement consisting of basic open sets.*

b. *If $A, B \subseteq X$ are closed and disjoint then*

$$\mathrm{Cl}_{X'}A \cap \mathrm{Cl}_{X'}B = \varnothing. \quad \square$$

The next result is from **[Ha]**.

2.1 LEMMA. *For all $n \in \mathbf{N}$: $(X')^n$ is homeomorphic to X', and the homeomorphism can be chosen to map X^n onto X.*

Now we are ready to prove the main result.

2.2 THEOREM. *The space X is both 2-fully normal and orthocompact.*

PROOF. Let $\mathcal{U}$ be a basic open cover of X. Put $U = \bigcup\{0 \times 0 \times 0: 0 \in \mathcal{U}\}$; U is a neighborhood of $\{\langle x, x, x\rangle: x \in X\}$ in X^3. Using 2.1 and 2.0b find a neighborhood U' of $\{\langle x, x, x\rangle: x \in X'\}$ in $(X')^3$ such that $U' \cap X^3 = U$.

For $x \in X' \setminus X$, choose $U_x \ni x$ open such that $U_x^3 \subseteq U'$.

By 2.0a let $\mathcal{O}'$ be a disjoint basic open refinement of the open cover

$$\{X' \setminus \mathrm{Cl}_{X'}(X \setminus 0)\}_{0 \in \mathcal{U}} \cup \{U_x\}_{x \in X' \setminus X}.$$

Let $\mathcal{O} = \{0' \cap X: 0' \in \mathcal{O}'\}$.

Let $0 \in \mathcal{O}$ and $\{x, y, z\} \subseteq 0$.

Then $\{x, y, z\} \subseteq$ some $V \in \mathcal{U}$ or $\{x, y, z\} \subseteq$ some U_p, but then $\langle x, y, z\rangle \in U_p^3 \cap X^3 \subseteq U$, so $\langle x, y, z\rangle \in V^3$ for some $V \in \mathcal{U}$ in any case. This implies that $\{x, y, z\} \subseteq V$.

For each $0 \in \mathcal{O}$ define $\mathcal{W}_0$ as follows: $0 = U_{p,q}$ for some $p, q \in F$, so put $\mathcal{W}_0 = \{U_{p,x}: x \in 0\}$. Let $\mathcal{W} = \bigcup\{\mathcal{W}_0: 0 \in \mathcal{O}\}$. Then $\mathcal{W}$ is both a 2-star and a Q-refinement of $\mathcal{U}$.

First, assume $U_{p,x} \cap U_{q,y} \neq \varnothing$ for some $U_{p,x}$ and $U_{q,y}$ in $\mathcal{W}$. Then x and y are elements of the same $0 \in \mathcal{O}$ and hence $p = q$. Define p' by $p'(n) = p(n) + \omega_1$ $(n \in \mathbf{N})$; then $p < p' \leqslant x, y$ and $p' \in X$, so $p' \in 0$.

Pick $u \in \mathcal{U}$ such that $\{p', x, y\} \subseteq U$. Since U is basic (and hence $\leqslant$-convex) and $U_{p,z} = \{t: p' \leqslant t \leqslant z\}$ for $z = x, y$, it follows that $U_{p,x} \cup U_{p,y} \subseteq U$. So $\mathcal{W}$ is a 2-star refinement of $\mathcal{U}$. Second, let $\mathcal{W}' \subseteq \mathcal{W}$ with $\bigcap \mathcal{W}' \neq \varnothing$. Then all $W \in \mathcal{W}'$ are contained in the same $0 \in \mathcal{O}$, so $\mathcal{W}' = \{U_{p,x}: x \in A\}$ for some subset A of 0, where $0 = U_{p,q}$. Define f by $f(n) = \min\{x(n): x \in A\}$. Then $\bigcap \mathcal{W}' = U_{p,f}$ is open. So $\mathcal{W}$ is a Q-refinement of $\mathcal{U}$. □

It now follows easily that X is finitely-fully normal:

2.3 COROLLARY. *X is finitely-fully normal.*

PROOF. Let $\mathcal{U}$ be an open cover of X. Let $\mathcal{V}_1$ be a 2-star refinement of $\mathcal{U}$, and (inductively) let $\mathcal{V}_{n+1}$ be a 2-star refinement of $\mathcal{V}_n$ ($n \in \mathbf{N}$). Since X is a P-space (G_δ's are open) we can take the common refinement of all $\mathcal{V}_n$; call it $\mathcal{V}$. Let $\mathcal{V}' \subseteq \mathcal{V}$ be finite with $\bigcap \mathcal{V}' \neq \varnothing$. Pick $n \in \mathbf{N}$ such that $2^n \geqslant |\mathcal{V}'|$. Since $\mathcal{V}$ refines $\mathcal{V}_n$ and since $\mathcal{V}_n$ is a 2^n-star refinement of $\mathcal{U}$, it follows that $\bigcup \mathcal{V}'$ is contained in some $U \in \mathcal{U}$. □

REFERENCES

[**Ha**] K. P. Hart, *Strong collectionwise normality and M. E. Rudin's Dowker space*, Proc. Amer. Math. Soc. **83** (1981), 802–806.

[**Ma**] M. J. Mansfield, *Some generalizations of full normality*, Trans. Amer. Math. Soc. **86** (1957), 489–505.

[**Ru**] M. E. Rudin, *A normal space X for which $X \times I$ is not normal*, Fund. Math. **73** (1971), 179–186.

[**Sc**] B. M. Scott, *Toward a product theory for orthocompactness*, Studies in Topology, Academic Press, New York, 1975, pp. 517–537.

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