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LOWER BOUNDS FOR THE UNKNOTTING NUMBERS OF CERTAIN TORUS KNOTS

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ABSTRACT. In this paper we shall show that the unknotting numbers of the $(l, 2kl \pm 1)$-torus knots are at least $(k(l^2-1)-2)/2$ for l odd and $(kl^2-2)/2$ for l even, where l is an integer greater than one and k is a positive integer.

1. Introduction. Unless otherwise stated all manifolds and maps are smooth and all knots and links are in S^3.

The *unknotting number* of a knot is the minimum number of crossings which must be changed to make the knot trivial. Let l and m be integers. The (l, m)-*torus link* is the link which lies on an unknotted torus and sweeps around it l times in the longitude and m times in the meridian. When l and m are relatively prime, it is a knot and called the (l, m)-*torus knot*. Milnor [**3**] conjectured that the unknotting number of a torus knot is equal to the genus of it. It is well known that the genus of the (l, m)-torus knot is equal to $(l-1)(m-1)/2$. It is not hard to see that the unknotting number of a torus knot is at most the genus of it. In this paper we shall show the following.

THEOREM A. *Let l be an integer greater than one and k a positive integer. Then the unknotting numbers of the $(l, 2kl \pm 1)$-torus knots are at least $(k(l^2-1)-2)/2$ if l is odd, and $(kl^2-2)/2$ if l is even.*

Murasugi [**4**] showed that the unknotting number of the $(2, m)$-torus knot is equal to the genus of it. Weintraub [**8**] showed that the unknotting number of the $(m-1, m)$-torus knot is at least $(m^2-5)/4$ if m is odd, and $(m^2-4)/4$ if m is even.

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2. Preliminaries. The following is a theorem of Rohlin [**5**], Hsiang-Szczarba [**9**], Thomas-Wood [**6**] and Weintraub [**7**].

THEOREM 1. *Let N be an oriented, connected, simply connected, closed 4-manifold. Let M be an oriented, connected, closed surface embedded in N. Suppose that M represents a 2-homology class $[M]$ of $H_2(N; Z)$ and that $[M]$ is divisible by a positive integer d in the free abelian group $H_2(N; Z)$. Let g_M be the genus of M. Then*

$$2g_M \geq \frac{[M]^2}{d^2}\frac{d^2-1}{2} - \operatorname{rank} H_2(N; Z) - \operatorname{signature} N$$

for d odd, and similarly for d even with $d^2/2$ instead of $(d^2-1)/2$, where $[M]^2$ is the self-intersection number of M.

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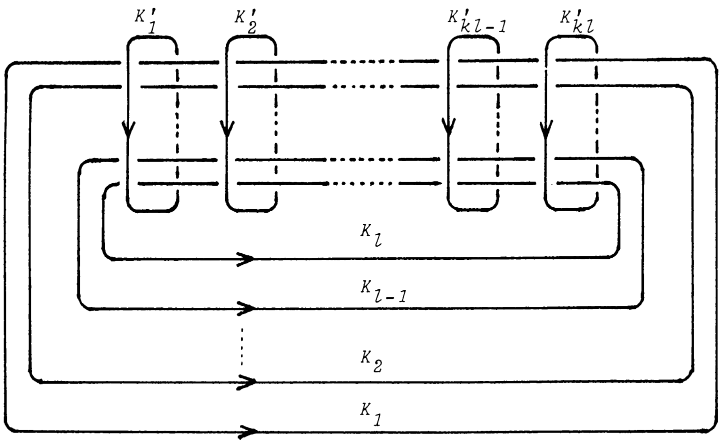


FIGURE 1

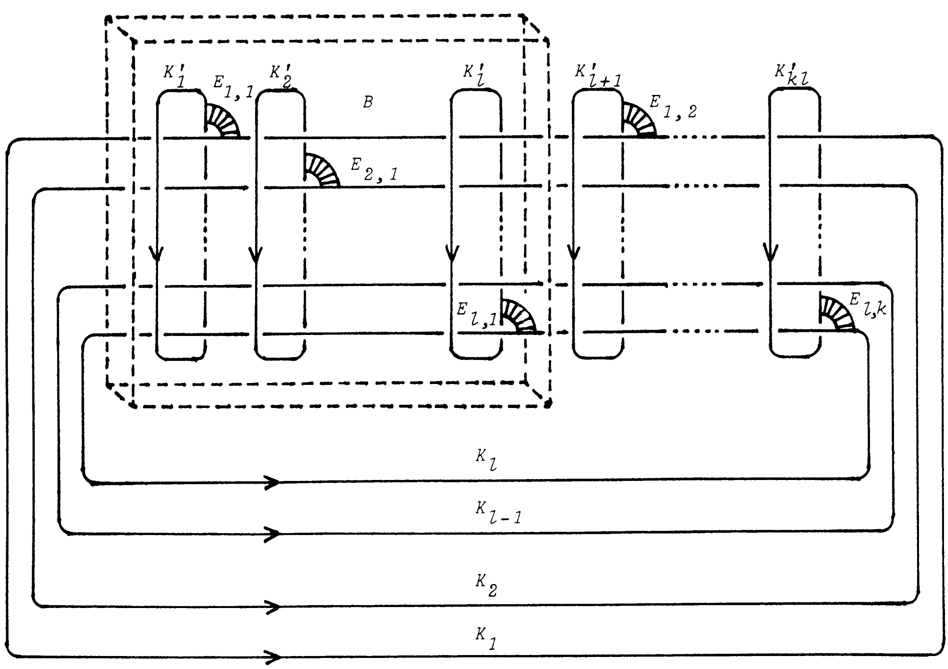


FIGURE 2

The following theorem originally due to Boardman [**1**] and Weintraub [**8**].

THEOREM 2. *Let N be an oriented 4-manifold, and let $N_0 = N - \operatorname{Int} B^4$, where B^4 is an embedded 4-ball. Suppose $\alpha \in H_2(N_0, \partial N_0; Z)$ is represented by an embedding $\Phi\colon (B^2, S^1) \to (N_0, \partial N_0)$ and let K denote the knot given by $\Phi|S^1\colon S^1 \to \partial N_0 = S^3$. If u is the unknotting number of K, then α is represented by an embedded 2-sphere in $N \# (uW)$, where $W = CP^2 \# (-CP^2)$.*

The proof is given by using W instead of CP^2 and $-CP^2$ of Theorem 7 of [**8**].

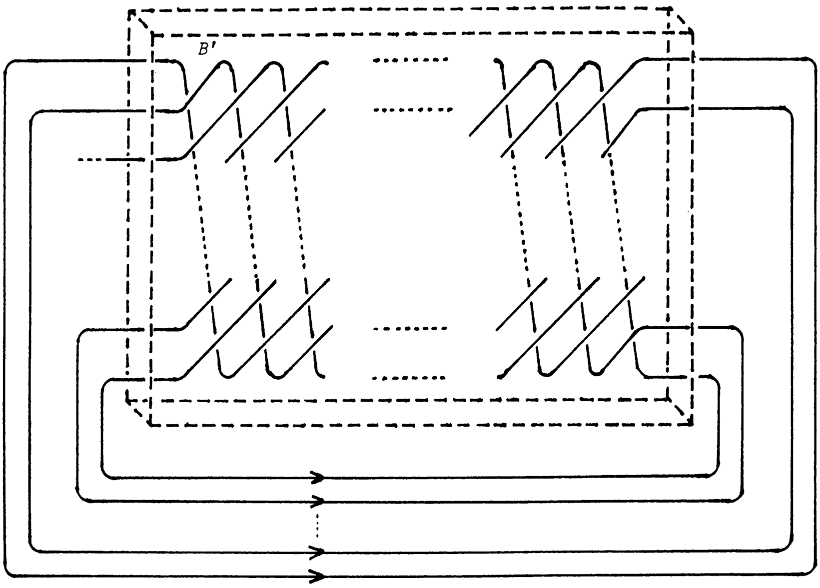


FIGURE 3

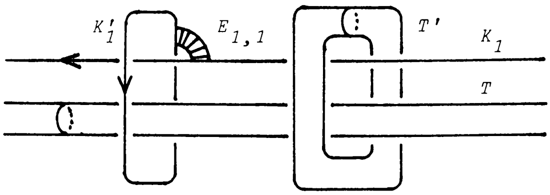


FIGURE 4

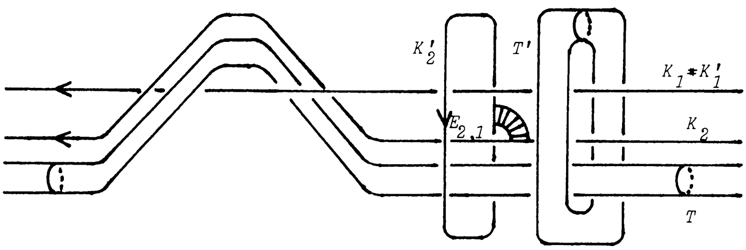


FIGURE 5

3. Proof of Theorem A. Let α and β be the generators of

$$H_2(S^2 \times S^2 - \operatorname{Int} B^4, \partial(S^2 \times S^2 - \operatorname{Int} B^4); Z),$$

and let l and m be positive integers. Then $l\alpha + m\beta$ can be represented by an embedded disk, say $D_{l,m}$. Let $K_{l,m}$ be the knot which is the boundary of $D_{l,m}$.

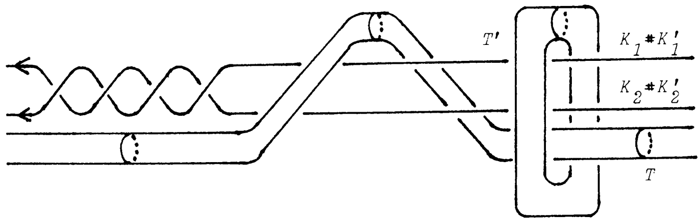


FIGURE 6

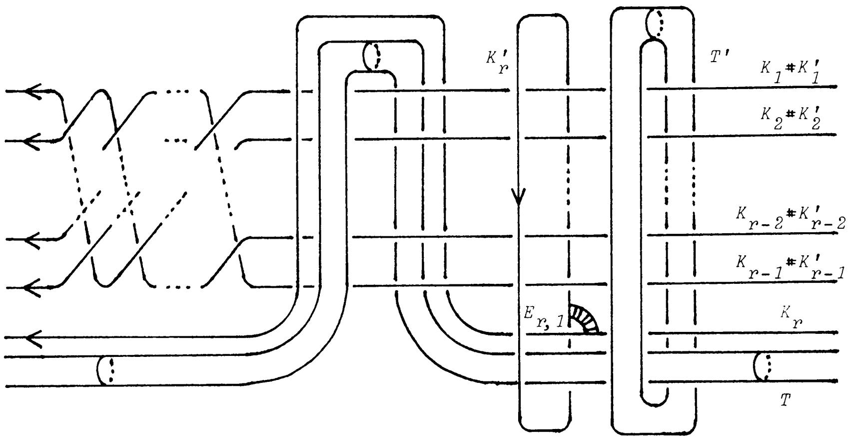


FIGURE 7

LEMMA. *Let u be the unknotting number of $K_{l,m}$. Suppose that l and m are divisible by a positive integer d. Then*

$$u \geq \begin{cases} \dfrac{lm(d^2-1)}{2d^2} - 1 & \textit{if } d \textit{ is odd}, \\ \dfrac{lm}{2} - 1 & \textit{if } d \textit{ is even}. \end{cases}$$

PROOF. By Theorem 2, $l\alpha + m\beta$ is represented by an embedded 2-sphere in $(S^2 \times S^2) \# (uW)$. Then, by Theorem 1,

$$0 \geq \frac{2lm}{d^2}\frac{d^2-1}{2} - 2(1+u) \quad \text{if } d \text{ is odd}$$

and

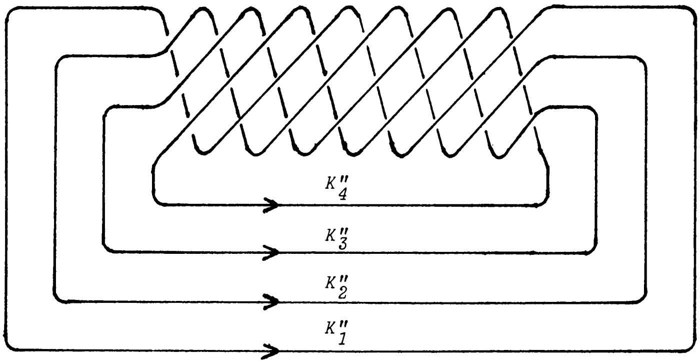
$$0 \geq \frac{2lm}{d^2}\frac{d^2}{2} - 2(1+u) \quad \text{if } d \text{ is even},$$

and elementary algebra yields the lemma.

Let l be an integer greater than one, and let k be a positive integer. We show that the class

$$l\alpha + kl\beta \in H_2(S^2 \times S^2 - \operatorname{Int} B^4, \partial(S^2 \times S^2 - \operatorname{Int} B^4); Z)$$

can be represented by an embedded disk with boundary the $(l, 2kl \pm 1)$-torus knot. The class $l\alpha + kl\beta$ can be represented by $l + kl$ embedded disks with boundary the link L of $l + kl$ components illustrated in Figure 1. Let $K_1, \dots, K_l$, $K_1', \dots, K_{kl}'$



The $(4, 8)$-torus link
FIGURE 8

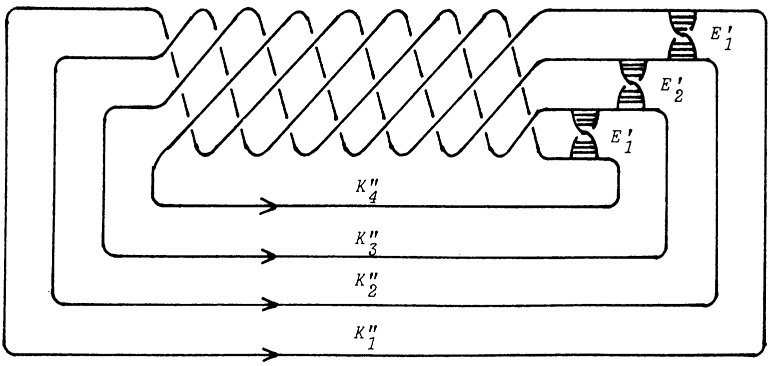


FIGURE 9

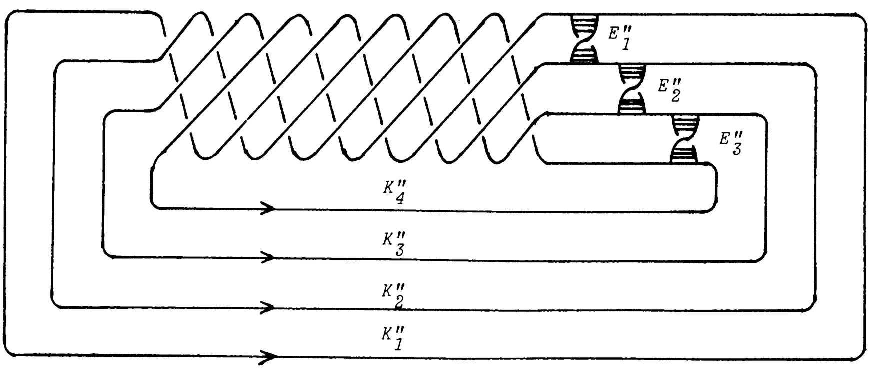


FIGURE 10

be as in Figure 1. Then $l\alpha + kl\beta$ can be represented by the disk obtained by connecting $K_1, \ldots, K_l, K'_1, \ldots, K'_{kl}$ by $l + kl - 1$ strips in $\partial(S^2 \times S^2 - \operatorname{Int} B^4) = S^3$ (see Kervaire–Milnor [**2**]). We connect K_i and $K'_{(j-1)l+i}$ by the strip $E_{i,j}$ as in Figure 2 for $i = 1, \ldots, l$ and $j = 1, \ldots, k$, thereby obtaining a collection of l disks representing the class $l\alpha + kl\beta$, whose boundaries form a link of l components.

Let L' be the link which consists of the boundaries of the disks obtained by the above construction. We show that the link L' is the $(l, 2kl)$-torus link as follows; it is sufficient to show that there is an isotopy of S^3 which deforms the part of L' contained in the 3-ball B in Figure 2 into the part of the $(l, 2l)$-torus link contained in the 3-ball B' in Figure 3 and which is relative to the complement of B. In Figures 4, 5, 6 and 7 we illustrate only the parts contained in the 3-balls, and all isotopies are relative to the complements of the 3-balls. We may consider that the components $K_2, \dots, K_l$ $(K'_2, \dots, K'_l)$ of L' are contained in a narrow tube T (resp. T') as in Figure 4. We connect K_1 and K'_1 by the strip $E_{1,1}$. Then we have the link which is isotopic to the link illustrated in Figure 5. We take K_2 (K'_2) out of T (resp. T') and connect K_2 and K'_2 by the strip $E_{2,1}$. Then we have the link which is isotopic to the link as in Figure 6, where the components $K_1 \# K'_1$ and $K_2 \# K'_2$ form the $(2, 4)$-torus link. We suppose that we have the link as in Figure 7 after connecting K_i and K'_i by the strip $E_{i,1}$ for $i = 1, \dots, r-1$ $(2 \leq r \leq l-1)$ as above and that the components $K_1 \# K'_1, K_2 \# K'_2, \dots, K_{r-1} \# K'_{r-1}$ form the $(r-1, 2(r-1))$-torus link. We connect K_r and K'_r by the strip $E_{r,1}$ as in Figure 7. Then we have the link which is isotopic to the similar link as in Figure 7 with $K_{r-1} \# K'_{r-1}$, $K_r \# K'_r$, K_{r+1} and K'_{r+1} instead of $K_{r-2} \# K'_{r-2}$, $K_{r-1} \# K'_{r-1}$, K_r and K'_r. Then we have inductively the link $(K_1 \# K'_1) \cup \cdots \cup (K_l \# K'_l)$ which is isotopic to the $(l, 2l)$-torus link by the composite of the above isotopies as required. Therefore we have shown that the link L' is the $(l, 2kl)$-torus link.

The $(l, 2kl)$-torus link is represented as in Figure 8. Let K''_i be the component $K_i \# K'_i \# K'_{l+i} \# \dots \# K'_{(k-1)l+i}$ of L'. We connect K''_i and K''_{i+1} $(i = 1, \dots, l-1)$ by the strip E'_i as in Figure 9. Then we have the $(l, 2kl-1)$-torus knot. When we connect K''_i and K''_{i+1} $(i = 1, \dots, l-1)$ by the strip E''_i as in Figure 10, we have the $(l, 2kl+1)$-torus knot. By Lemma, we obtain Theorem A.

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