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# LOWER BOUNDS FOR THE UNKNOTTING NUMBERS OF CERTAIN TORUS KNOTS

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ABSTRACT. In this paper we shall show that the unknotting numbers of the $(l, 2kl \pm 1)$-torus knots are at least $(k(l^2-1)-2)/2$ for $l$ odd and $(kl^2-2)/2$ for $l$ even, where $l$ is an integer greater than one and $k$ is a positive integer.

**1. Introduction.** Unless otherwise stated all manifolds and maps are smooth and all knots and links are in $S^3$.

The *unknotting number* of a knot is the minimum number of crossings which must be changed to make the knot trivial. Let $l$ and $m$ be integers. The $(l,m)$-*torus link* is the link which lies on an unknotted torus and sweeps around it $l$ times in the longitude and $m$ times in the meridian. When $l$ and $m$ are relatively prime, it is a knot and called the $(l,m)$-*torus knot*. Milnor [3] conjectured that the unknotting number of a torus knot is equal to the genus of it. It is well known that the genus of the $(l,m)$-torus knot is equal to $(l-1)(m-1)/2$. It is not hard to see that the unknotting number of a torus knot is at most the genus of it. In this paper we shall show the following.

THEOREM A. *Let $l$ be an integer greater than one and $k$ a positive integer. Then the unknotting numbers of the $(l, 2kl \pm 1)$-torus knots are at least $(k(l^2-1)-2)/2$ if $l$ is odd, and $(kl^2-2)/2$ if $l$ is even.*

Murasugi [4] showed that the unknotting number of the $(2,m)$-torus knot is equal to the genus of it. Weintraub [8] showed that the unknotting number of the $(m-1,m)$-torus knot is at least $(m^2-5)/4$ if $m$ is odd, and $(m^2-4)/4$ if $m$ is even.

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**2. Preliminaries.** The following is a theorem of Rohlin [5], Hsiang-Szczarba [9], Thomas-Wood [6] and Weintraub [7].

THEOREM 1. *Let $N$ be an oriented, connected, simply connected, closed 4-manifold. Let $M$ be an oriented, connected, closed surface embedded in $N$. Suppose that $M$ represents a 2-homology class $[M]$ of $H_2(N;Z)$ and that $[M]$ is divisible by a positive integer $d$ in the free abelian group $H_2(N;Z)$. Let $g_M$ be the genus of $M$. Then*

$$2g_M \geq \frac{[M]^2}{d^2}\frac{d^2-1}{2} - \operatorname{rank} H_2(N;Z) - \operatorname{signature} N$$

*for $d$ odd, and similarly for $d$ even with $d^2/2$ instead of $(d^2-1)/2$, where $[M]^2$ is the self-intersection number of $M$.*

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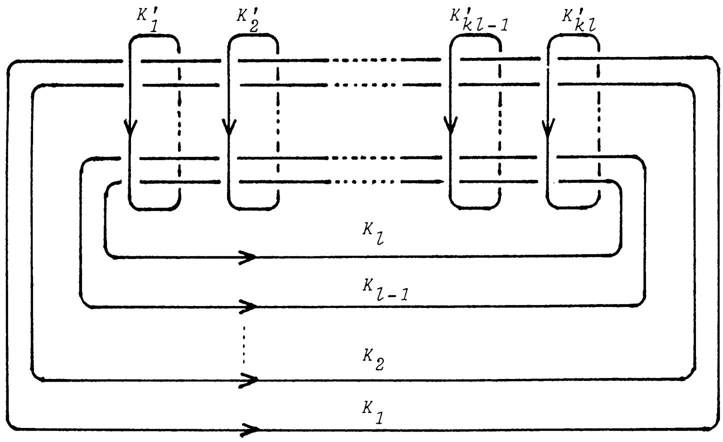



FIGURE 1

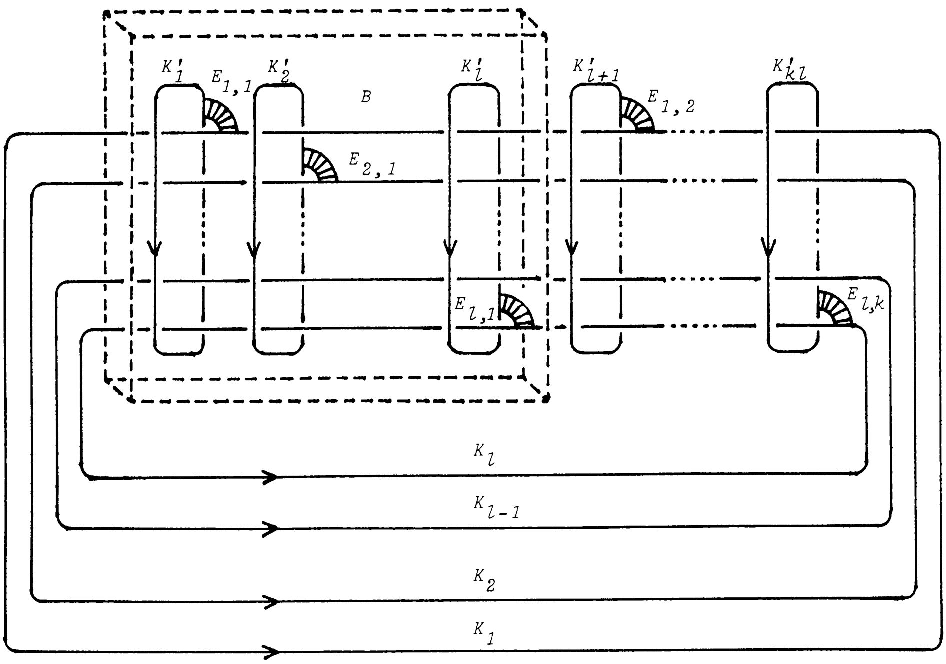



FIGURE 2

The following theorem originally due to Boardman [**1**] and Weintraub [**8**].

THEOREM 2. *Let $N$ be an oriented 4-manifold, and let $N_0 = N - \operatorname{Int} B^4$, where $B^4$ is an embedded 4-ball. Suppose $\alpha \in H_2(N_0, \partial N_0; Z)$ is represented by an embedding $\Phi\colon (B^2, S^1) \to (N_0, \partial N_0)$ and let $K$ denote the knot given by $\Phi|S^1\colon S^1 \to \partial N_0 = S^3$. If $u$ is the unknotting number of $K$, then $\alpha$ is represented by an embedded 2-sphere in $N \# (uW)$, where $W = CP^2 \# (-CP^2)$.*

The proof is given by using $W$ instead of $CP^2$ and $-CP^2$ of Theorem 7 of [**8**].

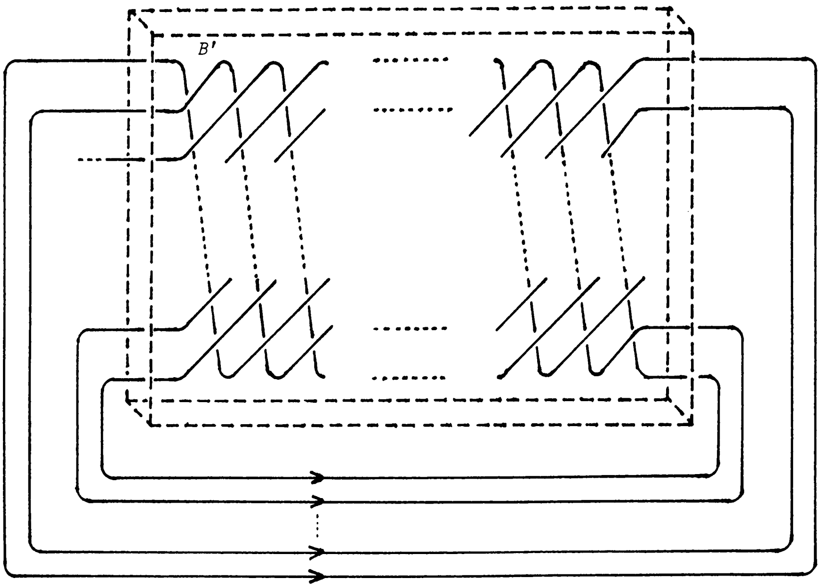



FIGURE 3

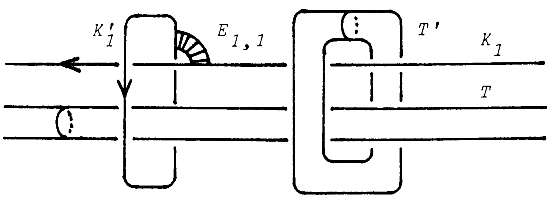



FIGURE 4

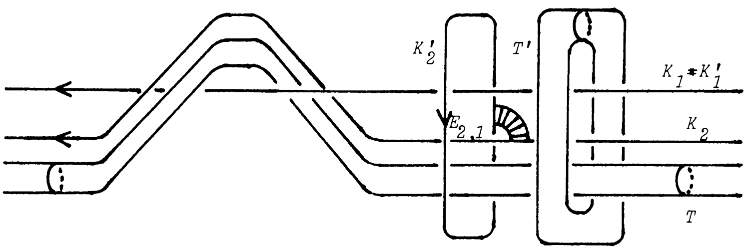



FIGURE 5

**3. Proof of Theorem A.** Let $\alpha$ and $\beta$ be the generators of

$$H_2(S^2 \times S^2 - \operatorname{Int} B^4, \partial(S^2 \times S^2 - \operatorname{Int} B^4); Z),$$

and let $l$ and $m$ be positive integers. Then $l\alpha + m\beta$ can be represented by an embedded disk, say $D_{l,m}$. Let $K_{l,m}$ be the knot which is the boundary of $D_{l,m}$.

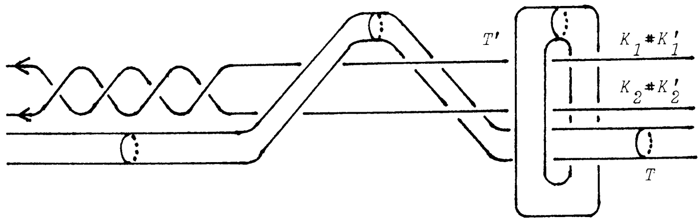



FIGURE 6

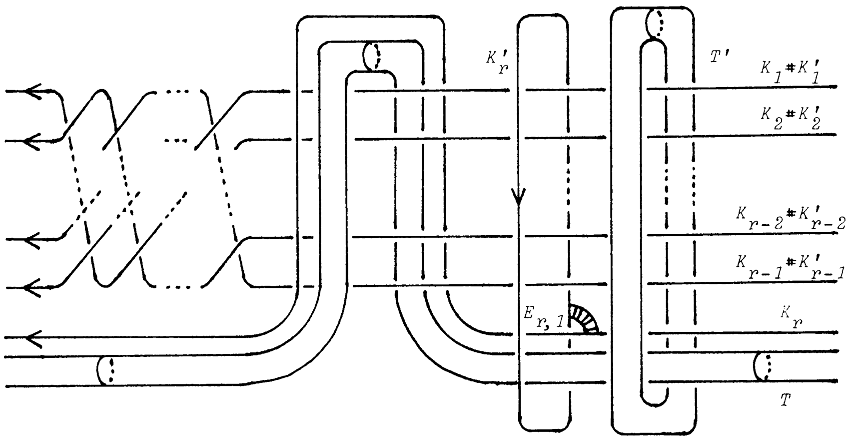



FIGURE 7

LEMMA. *Let $u$ be the unknotting number of $K_{l,m}$. Suppose that $l$ and $m$ are divisible by a positive integer $d$. Then*

$$u \geq \begin{cases} \dfrac{lm(d^2-1)}{2d^2} - 1 & \textit{if } d \textit{ is odd,} \\ \dfrac{lm}{2} - 1 & \textit{if } d \textit{ is even.} \end{cases}$$

PROOF. By Theorem 2, $l\alpha + m\beta$ is represented by an embedded 2-sphere in $(S^2 \times S^2) \# (uW)$. Then, by Theorem 1,

$$0 \geq \frac{2lm}{d^2} \frac{d^2-1}{2} - 2(1+u) \quad \text{if } d \text{ is odd}$$

and

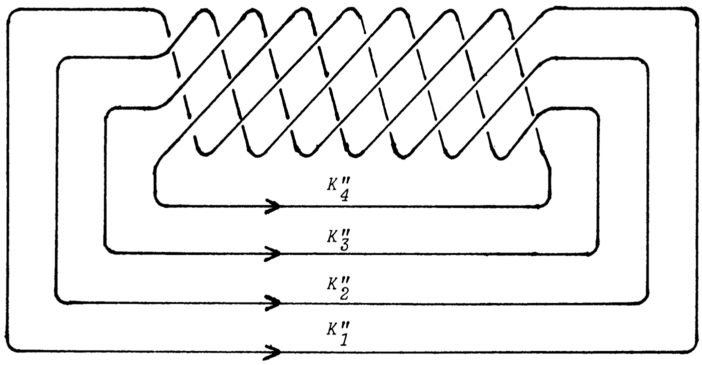
$$0 \geq \frac{2lm}{d^2} \frac{d^2}{2} - 2(1+u) \quad \text{if } d \text{ is even,}$$

and elementary algebra yields the lemma.

Let $l$ be an integer greater than one, and let $k$ be a positive integer. We show that the class

$$l\alpha + kl\beta \in H_2(S^2 \times S^2 - \operatorname{Int} B^4, \partial(S^2 \times S^2 - \operatorname{Int} B^4); Z)$$

can be represented by an embedded disk with boundary the $(l, 2kl \pm 1)$-torus knot. The class $l\alpha + kl\beta$ can be represented by $l + kl$ embedded disks with boundary the link $L$ of $l + kl$ components illustrated in Figure 1. Let $K_1, \ldots, K_l$, $K_1', \ldots, K_{kl}'$




The $(4,8)$-torus link

FIGURE 8

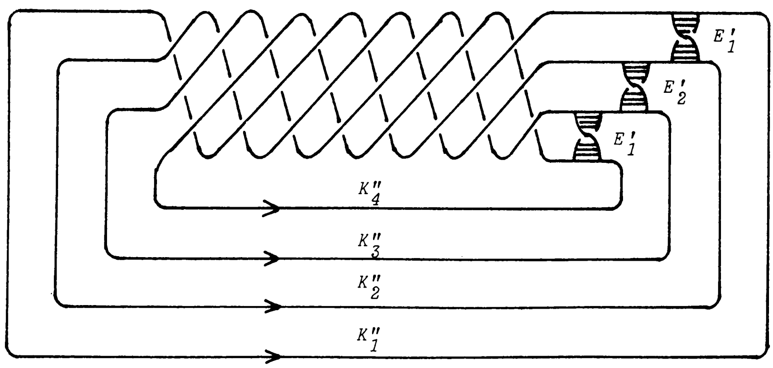



FIGURE 9

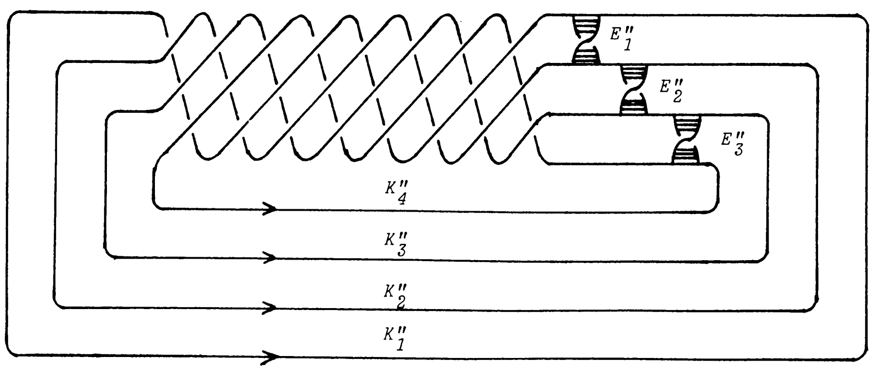



FIGURE 10

be as in Figure 1. Then $l\alpha + kl\beta$ can be represented by the disk obtained by connecting $K_1, \ldots, K_l, K'_1, \ldots, K'_{kl}$ by $l + kl - 1$ strips in $\partial(S^2 \times S^2 - \operatorname{Int} B^4) = S^3$ (see Kervaire–Milnor [**2**]). We connect $K_i$ and $K'_{(j-1)l+i}$ by the strip $E_{i,j}$ as in Figure 2 for $i = 1, \ldots, l$ and $j = 1, \ldots, k$, thereby obtaining a collection of $l$ disks representing the class $l\alpha + kl\beta$, whose boundaries form a link of $l$ components.

Let $L'$ be the link which consists of the boundaries of the disks obtained by the above construction. We show that the link $L'$ is the $(l, 2kl)$-torus link as follows; it is sufficient to show that there is an isotopy of $S^3$ which deforms the part of $L'$ contained in the 3-ball $B$ in Figure 2 into the part of the $(l, 2l)$-torus link contained in the 3-ball $B'$ in Figure 3 and which is relative to the complement of $B$. In Figures 4, 5, 6 and 7 we illustrate only the parts contained in the 3-balls, and all isotopies are relative to the complements of the 3-balls. We may consider that the components $K_2, \dots, K_l$ $(K'_2, \dots, K'_l)$ of $L'$ are contained in a narrow tube $T$ (resp. $T'$) as in Figure 4. We connect $K_1$ and $K'_1$ by the strip $E_{1,1}$. Then we have the link which is isotopic to the link illustrated in Figure 5. We take $K_2$ $(K'_2)$ out of $T$ (resp. $T'$) and connect $K_2$ and $K'_2$ by the strip $E_{2,1}$. Then we have the link which is isotopic to the link as in Figure 6, where the components $K_1 \# K'_1$ and $K_2 \# K'_2$ form the $(2, 4)$-torus link. We suppose that we have the link as in Figure 7 after connecting $K_i$ and $K'_i$ by the strip $E_{i,1}$ for $i = 1, \dots, r-1$ $(2 \le r \le l-1)$ as above and that the components $K_1 \# K'_1$, $K_2 \# K'_2, \dots, K_{r-1} \# K'_{r-1}$ form the $(r-1, 2(r-1))$-torus link. We connect $K_r$ and $K'_r$ by the strip $E_{r,1}$ as in Figure 7. Then we have the link which is isotopic to the similar link as in Figure 7 with $K_{r-1} \# K'_{r-1}$, $K_r \# K'_r$, $K_{r+1}$ and $K'_{r+1}$ instead of $K_{r-2} \# K'_{r-2}$, $K_{r-1} \# K'_{r-1}$, $K_r$ and $K'_r$. Then we have inductively the link $(K_1 \# K'_1) \cup \cdots \cup (K_l \# K'_l)$ which is isotopic to the $(l, 2l)$-torus link by the composite of the above isotopies as required. Therefore we have shown that the link $L'$ is the $(l, 2kl)$-torus link.

The $(l, 2kl)$-torus link is represented as in Figure 8. Let $K''_i$ be the component $K_i \# K'_i \# K'_{l+i} \# \dots \# K'_{(k-1)l+i}$ of $L'$. We connect $K''_i$ and $K''_{i+1}$ $(i = 1, \dots, l-1)$ by the strip $E'_i$ as in Figure 9. Then we have the $(l, 2kl-1)$-torus knot. When we connect $K''_i$ and $K''_{i+1}$ $(i = 1, \dots, l-1)$ by the strip $E''_i$ as in Figure 10, we have the $(l, 2kl+1)$-torus knot. By Lemma, we obtain Theorem A.

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