

FINITE 2-COMPLEXES WITH INFINITELY-GENERATED GROUPS OF SELF-HOMOTOPY-EQUIVALENCES

A. M. BRUNNER AND J. G. RATCLIFFE

ABSTRACT. Examples of finite 2-dimensional aspherical cell complexes are given whose group of homotopy classes of self-homotopy-equivalences is infinitely generated.

In a recent paper [11], Darryl McCullough gave an example of a finite, 4-dimensional, aspherical cell complex whose group of homotopy classes of self-homotopy-equivalences is infinitely generated, and asked if there is a 2-dimensional example. In this paper, we give an infinite family of finite, 2-dimensional, aspherical cell complexes each of whose group of homotopy classes of self-homotopy-equivalences is infinitely generated.

McCullough's example is based on Lewin's [8] example of a finitely presented group with an infinitely generated automorphism group. Our examples are based on recent examples, given by the first named author [3], of one-relator groups with infinitely-generated automorphism group.

Let X be a path connected space with base point. We let $\mathcal{E}(X)$ denote the group of based homotopy classes of self-homotopy-equivalences, and $\mathcal{F}(X)$ the group of free homotopy classes of self-homotopy-equivalences. If X is an aspherical CW-complex with fundamental group G , then $\mathcal{E}(X) \cong \text{Aut } G$ and $\mathcal{F}(X) \cong \text{Out } G$ [20, pp.225–226]. If G is finitely generated, then $\text{Aut } G$ is finitely generated if and only if $\text{Out } G$ is finitely generated. Thus, if X is a finite, aspherical complex, then $\mathcal{E}(X)$ is finitely generated if and only if $\mathcal{F}(X)$ is finitely generated.

EXAMPLE 1. Let n be a natural number greater than one, and consider the group

$$G(n) = \langle a, t; t^{-1}a^{-1}ta^nt^{-1}at = a \rangle.$$

The groups $\{G(n) \mid n = 2, 3, \dots\}$ are mutually nonisomorphic, since $G(n)$ abelianizes to $\mathbf{Z}_{n-1} \oplus \mathbf{Z}$. In [3], it is shown that $\text{Out } G(n) \cong \mathbf{Z}[\frac{1}{n}]$ —the additive group of rational numbers with denominator a nonnegative power of n . The group $\mathbf{Z}[\frac{1}{n}]$ is infinitely generated, since it is locally cyclic, but not cyclic. Thus $\text{Out } G(n)$ and $\text{Aut } G(n)$ are infinitely generated.

Let $X(n)$ be the 2-dimensional cell-complex which models the presentation of $G(n)$. Then $X(n)$ is a finite cell-complex (one 0-cell, two 1-cells and one 2-cell) with

Received by the editors March 6, 1981 and, in revised form, February 8, 1982.

1980 *Mathematics Subject Classification*. Primary 55P10; Secondary 57M20.

Key words and phrases. self-homotopy-equivalence, 2-dimensional complex, infinitely-generated group, aspherical complex, automorphism group, 1st k -invariant.

©1982 American Mathematical Society
0002-9939/82/0000-0439/\$02.50

fundamental group isomorphic to $G(n)$. As $X(n)$ has only one 2-cell, it follows from a lemma of Cockcroft [4] (see also [5]) that $X(n)$ is aspherical. Therefore, $\mathfrak{F}(X(n)) \cong \text{Out}(G(n)) \cong \mathbf{Z}[\frac{1}{n}]$. Thus $\mathfrak{F}(X(n))$ and $\mathfrak{E}(X(n))$ are not finitely generated.

At this point we would like to thank the referee for asking the following interesting question.

Question 1. Is there a finite two-dimensional complex X with $\text{Aut}(\pi_1(X))$ finitely-generated but $\mathfrak{E}(X)$ not finitely-generated?

To see what answering this question would entail, let $\mathfrak{E}^1(X)$ and $G(X)$ be the kernel and image respectively of the evaluation homomorphism

$$\mathfrak{E}(X) \rightarrow \text{Aut}(\pi_1(X)).$$

Then we have an exact sequence

$$1 \rightarrow \mathfrak{E}^1(X) \rightarrow \mathfrak{E}(X) \rightarrow G(X) \rightarrow 1.$$

If $\mathfrak{E}(X)$ is not finitely generated then either $G(X)$ or $\mathfrak{E}^1(X)$ is not finitely generated. This suggests Questions 2 and 3.

Question 2. Is there a finite two-dimensional complex X with $\text{Aut}(\pi_1(X))$ finitely-generated but $G(X)$ not finitely-generated?

This question appears to be difficult, since not much is known about the relation of $G(X)$ to $\text{Aut}(\pi_1(X))$. What is known is that if X is aspherical or if X has only one 2-cell (Jajodia [7]) then $G(X) = \text{Aut}(\pi_1(X))$, while Schellenberg [15] and Sieradski [18] have shown that $G(X)$ may be a proper subgroup of $\text{Aut}(\pi_1(X))$ when $\pi_1(X)$ is a noncyclic finite abelian group.

Question 3. Is there a finite 2-dimensional complex X such that $\mathfrak{E}^1(X)$ is not finitely generated?

Before we answer this question below, we need to analyze $\mathfrak{E}^1(X)$ when X is a finite 2-dimensional complex. Consider the evaluation homomorphism $\mathfrak{E}^1(X) \rightarrow \text{Aut}_{\pi_1}(\pi_2(X))$. It follows from equation (1) in Adams [1] that the image is the subgroup $\text{Aut}_{\pi_1}^k(\pi_2(X))$ of all π_1 -automorphisms of $\pi_2(X)$ which fix $k(X)$, the first k -invariant of X in $H^3(\pi_1(X), \pi_2(X))$; moreover, Schellenberg [14] has shown that the kernel is isomorphic to $H^2(\pi_1(X), \pi_2(X))$. Thus we have an exact sequence

$$0 \rightarrow H^2(\pi_1(X), \pi_2(X)) \rightarrow \mathfrak{E}^1(X) \rightarrow \text{Aut}_{\pi_1}^k(\pi_2(X)) \rightarrow 1.$$

In order to compute $\mathfrak{E}^1(X)$, it would be convenient to have $k(X) = 0$, since then $\text{Aut}_{\pi_1}^k(\pi_2(X)) = \text{Aut}_{\pi_1}(\pi_2(X))$. This certainly is the case if the cohomological dimension of $\pi_1(X)$ is at most two, since $k(X)$ lives in a 3-dimensional cohomology group of $\pi_1(X)$. The following theorem says that, in fact, the converse is true.

THEOREM 1. *Let $k(X)$ be the first k -invariant of a 2-dimensional CW-complex X . Then $k(X) = 0$ if and only if the cohomological dimension of $\pi_1(X)$ is at most two.*

PROOF. Suppose $k(X) = 0$. Recall that $k(X)$ is represented by the crossed sequence [10]

$$0 \rightarrow \pi_2(X) \rightarrow \pi_2(X, X^1) \xrightarrow{\partial_2} \pi_1(X^1) \rightarrow \pi_1(X) \rightarrow 1$$

where X^1 is the 1-skeleton of X . Let $N = \text{Im } \partial_2$. Then N is the group of relations of the presentation for $\pi_1(X)$ prescribed by the cell structure of X . By Theorem 8.1 of [12], the crossed extension

$$0 \rightarrow \pi_2(X) \rightarrow \pi_2(X, X^1) \rightarrow N \rightarrow 1$$

operator splits. As $\pi_2(X, X^1)$ is a free crossed module, we have that N is a projective crossed module [13]. This implies that the relation module N_{ab} is a projective $\pi_1(X)$ -module. Therefore, $\text{cd } \pi_1(X) \leq 2$ by Serre [17, p. 90]. \square

Next, we give an example which may possibly give an affirmative answer to Question 1.

EXAMPLE 2. Let $X = S^1 \vee S^2 \vee S^2$. Then $\pi_1(X) \cong \mathbf{Z}$ and $\pi_2(X) \cong \mathbf{Z}(\mathbf{Z}) \oplus \mathbf{Z}(\mathbf{Z})$. Clearly, $G(X) = \text{Aut}(\mathbf{Z}) = \pm 1$. As $\text{cd } \mathbf{Z} = 1$, we have that $H^i(\pi_1(X), \pi_2(X)) = 0$ for $i = 2, 3$; therefore, $\mathcal{E}^1(X) \cong \text{Aut}_{\pi_1}(\pi_2(X)) \cong \text{GL}_2(\mathbf{Z}(\mathbf{Z}))$. Thus we have an exact sequence

$$1 \rightarrow \text{GL}_2(\mathbf{Z}(\mathbf{Z})) \rightarrow \mathcal{E}(X) \rightarrow \pm 1 \rightarrow 1.$$

In particular, $\mathcal{E}(X)$ is finitely generated if and only if $\text{GL}_2(\mathbf{Z}(\mathbf{Z}))$ is finitely generated. Note that $\mathbf{Z}(\mathbf{Z})$ is just the ring $\mathbf{Z}[t, t^{-1}]$ of Laurent polynomials in one variable t . Surprisingly, it is unknown whether or not $\text{GL}_2(\mathbf{Z}[t, t^{-1}])$ is finitely generated. Thus we ask

Question 4. Is the group $\text{GL}_2(\mathbf{Z}[t, t^{-1}])$ infinitely generated?

Note that an affirmative answer to Question 4 and Example 2 gives an affirmative answer to Question 1. Also, see Lemma 1 below.

Let $\mathcal{E}^2(X)$ be the kernel of the evaluation homomorphism

$$\mathcal{E}^1(X) \rightarrow \text{Aut}_{\pi_1}(\pi_2(X)).$$

Then $\mathcal{E}^2(X)$ is the group of homotopy classes of self-homotopy-equivalences of X which induce the identity on both $\pi_1(X)$ and $\pi_2(X)$. Schellenberg remarked in [14] that he knew of no example of a finite 2-complex X such that $\mathcal{E}^2(X)$ is nontrivial. Next, we give two such examples, one for which $\mathcal{E}^2(X)$ is finitely generated and the other for which $\mathcal{E}^2(X)$ is infinitely generated.

EXAMPLE 3. Let $X = T^2 \vee S^2$ where T^2 is the torus. Then $\pi_1(X) \cong \mathbf{Z} \oplus \mathbf{Z}$ and $\pi_2(X) \cong \mathbf{Z}(\pi_1(X))$. By Poincaré duality

$$H^2(\pi_1, \pi_2) \cong H_0(\pi_1, \pi_2) \cong \mathbf{Z}.$$

Therefore, $\mathcal{E}^2(X)$ is infinite cyclic. Marshall Cohen described to us a map representing a generator of $\mathcal{E}^2(X)$ as gotten by pinching a 2-sphere off T^2 at the base point and mapping it by a degree one map to the wedged S^2 . It is interesting to note that this map induces a nontrivial automorphism of $H_2(X)$.

To complete the analysis of $\mathcal{E}(X)$, notice that the first k -invariant of X is zero since $\mathbf{Z} \oplus \mathbf{Z}$ is a 2-dimensional group. Therefore, $\text{Aut}_{\pi_1}^k(\pi_2(X))$ is isomorphic to the group of units of $\mathbf{Z}(\pi_1(X))$. By Theorem 13 of Higman [6] the ring $\mathbf{Z}(\pi_1(X))$ has only trivial units. Thus $\mathcal{E}(X)$ is determined up to extension by the exact sequences

$$\begin{aligned} 1 &\rightarrow \mathcal{E}^1(X) \rightarrow \mathcal{E}(X) \rightarrow \text{GL}_2(\mathbf{Z}) \rightarrow 1, \\ 0 &\rightarrow \mathbf{Z} \rightarrow \mathcal{E}^1(X) \rightarrow \pm \mathbf{Z} \oplus \mathbf{Z} \rightarrow 0. \end{aligned}$$

In particular, $\mathcal{E}^2(X)$, $\mathcal{E}^1(X)$ and $\mathcal{E}(X)$ are all finitely generated.

EXAMPLE 4. Let $X = K \vee S^2$ where K is the finite 2-complex which models the presentation $(x, y; x^2 = y^3)$ of the trefoil knot group G . Then $\pi_1(X) \cong G$, $\pi_2(X) \cong \mathbf{Z}(G)$ and $\mathcal{E}^2(X) \cong H^2(G; \mathbf{Z}(G))$.

The center of G is generated by x^2 ; therefore, we have a central extension

$$0 \rightarrow \mathbf{Z} \rightarrow G \rightarrow Q \rightarrow 1$$

where $Q = \mathbf{Z}_2 * \mathbf{Z}_3$.

Next, feed this extension into the Lyndon-Hochschild-Serre spectral sequence to compute $H^2(G, \mathbf{Z}(G))$. One computes easily that $H^2(G, \mathbf{Z}(G)) \cong E_2^{1,1}$. By Poincaré duality we have

$$H^1(\mathbf{Z}, \mathbf{Z}(G)) \cong H_0(\mathbf{Z}, \mathbf{Z}(G)) \cong \mathbf{Z}(Q).$$

Therefore, $H^2(G, \mathbf{Z}(G)) \cong H^1(Q, \mathbf{Z}(Q))$. It is well known [19] that $H^1(Q, \mathbf{Z}(Q))$ is a free abelian group of countably infinite rank. Therefore, $\mathcal{E}^2(X)$ is a free abelian group of countably infinite rank.

The commutator subgroup of G is free of rank 2 and $G_{\text{ab}} \cong \mathbf{Z}$; therefore, G is indicable throughout by the lemma in the appendix of [6]. Hence, $\mathbf{Z}(G)$ has only trivial units. Thus, we have an exact sequence

$$0 \rightarrow \mathcal{E}^2(X) \rightarrow \mathcal{E}^1(X) \rightarrow \pm G \rightarrow 1.$$

At this point the reader should be wondering whether or not $\mathcal{E}^1(X)$ is infinitely generated. From the cochain complex obtained from the Lyndon resolution [9] of the one-relator group G , one sees that $H^2(G, \mathbf{Z}(G))$ is a cyclic right G -module; moreover, one sees from the description of $\mathcal{E}^1(X)$ in [14] that the action of $\pm G$ on $\mathcal{E}^2(X)$ induced by conjugation in $\mathcal{E}^1(X)$ corresponds to right translation in $H^2(G, \mathbf{Z}(G))$. This implies easily that $\mathcal{E}^1(X)$ is finitely generated.

Finally, we have an exact sequence

$$1 \rightarrow \mathcal{E}^1(X) \rightarrow \mathcal{E}(X) \rightarrow \text{Aut } G \rightarrow 1.$$

The group $\text{Aut } G$ is finitely generated [16]; therefore, $\mathcal{E}(X)$ is finitely generated.

Before we give our last example, which answers Question 3, we need to prove the following lemma.

LEMMA 1. *The group $\text{GL}_2(\mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}])$ is infinitely generated if and only if*

$$\text{SL}_2(\mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}])$$

is infinitely generated.

PROOF. Let $G = \text{GL}_2(\mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}])$, $G_1 = \text{SL}_2(\mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}])$, and $U = \mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]^*$. Then $U \cong \mathbf{Z}_2 \oplus \mathbf{Z}^n$ is generated multiplicatively by $-1, t_1, \dots, t_n$. Because of the exact sequence

$$1 \rightarrow G_1 \rightarrow G \xrightarrow{\text{Det}} U \rightarrow 1$$

the group G is infinitely generated only if G_1 is infinitely generated.

Conversely, suppose G_1 is infinitely generated. Let G_2 be the subgroup of G consisting of the matrices whose determinant is in the group generated by t_1^2, \dots, t_n^2 .

Then there is a retraction $\rho: G_2 \rightarrow G_1$ defined by

$$\rho(A) = \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} A \quad \text{where } \det A = u^2.$$

Therefore, G_2 is infinitely generated. But G_2 is a subgroup of index 2^{n+1} in G ; therefore, G is infinitely generated. \square

EXAMPLE 5. Let $X = T^2 \vee S^2 \vee S^2$. Then $\pi_1(X) \cong \mathbf{Z} \oplus \mathbf{Z}$ and $\pi_2(X) \cong \mathbf{Z}(\mathbf{Z} \oplus \mathbf{Z}) \oplus \mathbf{Z}(\mathbf{Z} \oplus \mathbf{Z})$. Therefore, we have an exact sequence

$$1 \rightarrow \mathfrak{S}^1(X) \rightarrow \mathfrak{S}(X) \rightarrow \mathrm{GL}_2(\mathbf{Z}) \rightarrow 1.$$

By Poincaré duality

$$H^2(\pi_1, \pi_2) \cong H_0(\pi_1, \pi_2) \cong \mathbf{Z} \oplus \mathbf{Z}.$$

Therefore, we have an exact sequence

$$0 \rightarrow \mathbf{Z} \oplus \mathbf{Z} \rightarrow \mathfrak{S}^1(X) \rightarrow \mathrm{GL}_2(\mathbf{Z}(\mathbf{Z} \oplus \mathbf{Z})) \rightarrow 1.$$

The ring $\mathbf{Z}(\mathbf{Z} \oplus \mathbf{Z})$ is just $\mathbf{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}]$.

Recently, Bachmuth and Mochizuki [2] have shown that $\mathrm{SL}_2(\mathbf{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}])$ is infinitely generated. Therefore, by the lemma, $\mathrm{GL}_2(\mathbf{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}])$ is infinitely generated. This implies that $\mathfrak{S}^1(X)$ is infinitely generated. However, it is unclear whether or not $\mathfrak{S}(X)$ is finitely generated, since $\mathrm{GL}_2(\mathbf{Z})$ is finitely generated.

Question 5. Is $\mathfrak{S}(T^2 \vee S^2 \vee S^2)$ infinitely generated?

REFERENCES

1. J. F. Adams, *Four applications of the self-obstruction invariants*, J. London Math. Soc. **31** (1956), 148–159.
2. S. Bachmuth and H. Y. Mochizuki, $E_2 \neq \mathrm{SL}_2$ for most Laurent polynomial rings, Preprint, Univ. of California at Santa Barbara, 1981.
3. A. M. Brunner, *On a class of one-relator groups*, Canad. J. Math. **32** (1980), 414–420.
4. W. H. Cockcroft, *On two-dimensional aspherical complexes*, Proc. London Math. Soc. **4** (1954), 375–384.
5. E. Dyer and A. T. Vasquez, *Some small aspherical spaces*, J. Austral. Math. Soc. **16** (1973), 332–352.
6. G. Higman, *The units of group-rings*, Proc. London Math. Soc. **46** (1940), 231–248.
7. S. Jajodia, *On 2-dimensional CW-complexes with a single 2-cell*, Pacific J. Math. **80** (1979), 191–203.
8. J. Lewin, *A finitely presented group whose group of automorphisms is infinitely generated*, J. London Math. Soc. **42** (1967), 610–613.
9. R. C. Lyndon, *Cohomology theory of groups with a single defining relation*, Ann. of Math. **52** (1950), 650–665.
10. S. Mac Lane and J. H. C. Whitehead, *On the 3-type of a complex*, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 41–48.
11. D. McCullough, *Finite aspherical complexes with infinitely-generated groups of self-homotopy-equivalences*, Proc. Amer. Math. Soc. **80** (1980), 337–340.
12. J. G. Ratcliffe, *Crossed extensions*, Trans. Amer. Math. Soc. **257** (1980), 73–89.
13. _____, *Free and projective crossed modules*, J. London Math. Soc. **22** (1980), 66–74.
14. B. Schellenberg, *The group of homotopy self-equivalences of some compact CW-complexes*, Math. Ann. **200** (1973), 253–266.
15. _____, *On the self-equivalences of a space with non-cyclic fundamental group*, Math. Ann. **205** (1973), 333–344.
16. O. Schreier, *Über die Gruppen $A^a B^b = 1$* , Abh. Math. Sem. Univ. Hamburg **3** (1924), 167–169.
17. J.-P. Serre, *Cohomologie des groupes discrets*, Prospects in Mathematics, Ann. of Math. Studies, no. 70, Princeton Univ. Press, Princeton, N. J., pp. 77–132.

18. A. J. Sieradski, *Combinatorial isomorphisms and combinatorial homotopy equivalences*, J. Pure Appl. Algebra 7 (1976), 59–95.
19. E. Specker, *Die erste Cohomologiegruppe von Überlagerungen und Homotopie-Eigenschaften dreidimensionaler Mannigfaltigkeiten*, Comment. Math. Helv. 23 (1949), 303–333.
20. G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Math., vol. 61, Springer-Verlag, New York, Heidelberg and Berlin, 1978.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-PARKSIDE, KENOSHA, WISCONSIN 53140

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

SCHOOL OF MATHEMATICS, THE INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540