PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 3, November 1982

FINITE 2-COMPLEXES WITH INFINITELY-GENERATED GROUPS OF SELF-HOMOTOPY-EQUIVALENCES

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ABSTRACT. Examples of finite 2-dimensional aspherical cell complexes are given whose group of homotopy classes of self-homotopy-equivalences is infinitely generated.

In a recent paper [**11**], Darryl McCullough gave an example of a finite, 4-dimensional, aspherical cell complex whose group of homotopy classes of self-homotopy-equivalences is infinitely generated, and asked if there is a 2-dimensional example. In this paper, we give an infinite family of finite, 2-dimensional, aspherical cell complexes each of whose group of homotopy classes of self-homotopy-equivalences is infinitely generated.

McCullough's example is based on Lewin's [**8**] example of a finitely presented group with an infinitely generated automorphism group. Our examples are based on recent examples, given by the first named author [**3**], of one-relator groups with infinitely-generated automorphism group.

Let X be a path connected space with base point. We let $\mathcal{E}(X)$ denote the group of based homotopy classes of self-homotopy-equivalences, and $\mathcal{F}(X)$ the group of free homotopy classes of self-homotopy-equivalences. If X is an aspherical CW-complex with fundamental group G, then $\mathcal{E}(X) \cong \operatorname{Aut} G$ and $\mathcal{F}(X) \cong \operatorname{Out} G$ [**20**, pp. 225–226]. If G is finitely generated, then $\operatorname{Aut} G$ is finitely generated if and only if $\operatorname{Out} G$ is finitely generated. Thus, if X is a finite, aspherical complex, then $\mathcal{E}(X)$ is finitely generated if and only if $\mathcal{F}(X)$ is finitely generated.

EXAMPLE 1. Let n be a natural number greater than one, and consider the group

$$G(n) = (a, t;\, t^{-1}a^{-1}ta^n t^{-1}at = a).$$

The groups $\{G(n) \mid n = 2, 3, \dots\}$ are mutually nonisomorphic, since $G(n)$ abelianizes to $\mathbf{Z}_{n-1} \oplus \mathbf{Z}$. In [**3**], it is shown that $\operatorname{Out} G(n) \cong \mathbf{Z}[\frac{1}{n}]$—the additive group of rational numbers with denominator a nonnegative power of n. The group $\mathbf{Z}[\frac{1}{n}]$ is infinitely generated, since it is locally cyclic, but not cyclic. Thus $\operatorname{Out} G(n)$ and $\operatorname{Aut} G(n)$ are infinitely generated.

Let $X(n)$ be the 2-dimensional cell-complex which models the presentation of $G(n)$. Then $X(n)$ is a finite cell-complex (one 0-cell, two 1-cells and one 2-cell) with

Received by the editors March 6, 1981 and, in revised form, February 8, 1982.
1980 *Mathematics Subject Classification*. Primary 55P10; Secondary 57M20.
Key words and phrases. self-homotopy-equivalence, 2-dimensional complex, infinitely-generated group, aspherical complex, automorphism group, 1st k-invariant.

fundamental group isomorphic to $G(n)$. As $X(n)$ has only one 2-cell, it follows from a lemma of Cockcroft [**4**] (see also [**5**]) that $X(n)$ is aspherical. Therefore, $\mathfrak{F}(X(n)) \cong \mathrm{Out}(G(n)) \cong \mathbf{Z}[\frac{1}{n}]$. Thus $\mathfrak{F}(X(n))$ and $\mathfrak{E}(X(n))$ are not finitely generated.

At this point we would like to thank the referee for asking the following interesting question.

Question 1. Is there a finite two-dimensional complex X with $\mathrm{Aut}(\pi_1(X))$ finitely-generated but $\mathfrak{E}(X)$ not finitely-generated?

To see what answering this question would entail, let $\mathfrak{E}^1(X)$ and $G(X)$ be the kernel and image respectively of the evaluation homomorphism

$$\mathfrak{E}(X) \to \mathrm{Aut}(\pi_1(X)).$$

Then we have an exact sequence

$$1 \to \mathfrak{E}^1(X) \to \mathfrak{E}(X) \to G(X) \to 1.$$

If $\mathfrak{E}(X)$ is not finitely generated then either $G(X)$ or $\mathfrak{E}^1(X)$ is not finitely generated. This suggests Questions 2 and 3.

Question 2. Is there a finite two-dimensional complex X with $\mathrm{Aut}(\pi_1(X))$ finitely-generated but $G(X)$ not finitely-generated?

This question appears to be difficult, since not much is known about the relation of $G(X)$ to $\mathrm{Aut}(\pi_1(X))$. What is known is that if X is aspherical or if X has only one 2-cell (Jajodia [**7**]) then $G(X) = \mathrm{Aut}(\pi_1(X))$, while Schellenberg [**15**] and Sieradski [**18**] have shown that $G(X)$ may be a proper subgroup of $\mathrm{Aut}(\pi_1(X))$ when $\pi_1(X)$ is a noncyclic finite abelian group.

Question 3. Is there a finite 2-dimensional complex X such that $\mathfrak{E}^1(X)$ is not finitely generated?

Before we answer this question below, we need to analyze $\mathfrak{E}^1(X)$ when X is a finite 2-dimensional complex. Consider the evaluation homomorphism $\mathfrak{E}^1(X) \to \mathrm{Aut}_{\pi_1}(\pi_2(X))$. It follows from equation (1) in Adams [**1**] that the image is the subgroup $\mathrm{Aut}^k_{\pi_1}(\pi_2(X))$ of all π_1-automorphisms of $\pi_2(X)$ which fix $k(X)$, the first k-invariant of X in $H^3(\pi_1(X), \pi_2(X))$; moreover, Schellenberg [**14**] has shown that the kernel is isomorphic to $H^2(\pi_1(X), \pi_2(X))$. Thus we have an exact sequence

$$0 \to H^2(\pi_1(X), \pi_2(X)) \to \mathfrak{E}^1(X) \to \mathrm{Aut}^k_{\pi_1}(\pi_2(X)) \to 1.$$

In order to compute $\mathfrak{E}^1(X)$, it would be convenient to have $k(X) = 0$, since then $\mathrm{Aut}^k_{\pi_1}(\pi_2(X)) = \mathrm{Aut}_{\pi_1}(\pi_2(X))$. This certainly is the case if the cohomological dimension of $\pi_1(X)$ is at most two, since $k(X)$ lives in a 3-dimensional cohomology group of $\pi_1(X)$. The following theorem says that, in fact, the converse is true.

THEOREM 1. *Let $k(X)$ be the first k-invariant of a 2-dimensional CW-complex X. Then $k(X) = 0$ if and only if the cohomological dimension of $\pi_1(X)$ is at most two.*

PROOF. Suppose $k(X) = 0$. Recall that $k(X)$ is represented by the crossed sequence [**10**]

$$0 \to \pi_2(X) \to \pi_2(X, X^1) \xrightarrow{\partial_2} \pi_1(X^1) \to \pi_1(X) \to 1$$

where X^1 is the 1-skeleton of X. Let $N = \operatorname{Im} \partial_2$. Then N is the group of relations of the presentation for $\pi_1(X)$ prescribed by the cell structure of X. By Theorem 8.1 of [**12**], the crossed extension

$$0 \to \pi_2(X) \to \pi_2(X, X^1) \to N \to 1$$

operator splits. As $\pi_2(X, X^1)$ is a free crossed module, we have that N is a projective crossed module [**13**]. This implies that the relation module N_{ab} is a projective $\pi_1(X)$-module. Therefore, $\operatorname{cd} \pi_1(X) \leqslant 2$ by Serre [**17**, p. 90]. □

Next, we give an example which may possibly give an affirmative answer to Question 1.

EXAMPLE 2. Let $X = S^1 \vee S^2 \vee S^2$. Then $\pi_1(X) \cong \mathbf{Z}$ and $\pi_2(X) \cong \mathbf{Z}(\mathbf{Z}) \oplus \mathbf{Z}(\mathbf{Z})$. Clearly, $G(X) = \operatorname{Aut}(\mathbf{Z}) = \pm 1$. As $\operatorname{cd} \mathbf{Z} = 1$, we have that $H^i(\pi_1(X), \pi_2(X)) = 0$ for $i = 2, 3$; therefore, $\mathcal{E}^1(X) \cong \operatorname{Aut}_{\pi_1}(\pi_2(X)) \cong \mathrm{GL}_2(\mathbf{Z}(\mathbf{Z}))$. Thus we have an exact sequence

$$1 \to \mathrm{GL}_2(\mathbf{Z}(\mathbf{Z})) \to \mathcal{E}(X) \to \pm 1 \to 1.$$

In particular, $\mathcal{E}(X)$ is finitely generated if and only if $\mathrm{GL}_2(\mathbf{Z}(\mathbf{Z}))$ is finitely generated. Note that $\mathbf{Z}(\mathbf{Z})$ is just the ring $\mathbf{Z}[t, t^{-1}]$ of Laurent polynomials in one variable t. Surprisingly, it is unknown whether or not $\mathrm{GL}_2(\mathbf{Z}[t, t^{-1}])$ is finitely generated. Thus we ask

Question 4. Is the group $\mathrm{GL}_2(\mathbf{Z}[t, t^{-1}])$ infinitely generated?

Note that an affirmative answer to Question 4 and Example 2 gives an affirmative answer to Question 1. Also, see Lemma 1 below.

Let $\mathcal{E}^2(X)$ be the kernel of the evaluation homomorphism

$$\mathcal{E}^1(X) \to \operatorname{Aut}_{\pi_1}(\pi_2(X)).$$

Then $\mathcal{E}^2(X)$ is the group of homotopy classes of self-homotopy-equivalences of X which induce the identity on both $\pi_1(X)$ and $\pi_2(X)$. Schellenberg remarked in [**14**] that he knew of no example of a finite 2-complex X such that $\mathcal{E}^2(X)$ is nontrivial. Next, we give two such examples, one for which $\mathcal{E}^2(X)$ is finitely generated and the other for which $\mathcal{E}^2(X)$ is infinitely generated.

EXAMPLE 3. Let $X = T^2 \vee S^2$ where T^2 is the torus. Then $\pi_1(X) \cong \mathbf{Z} \oplus \mathbf{Z}$ and $\pi_2(X) \cong \mathbf{Z}(\pi_1(X))$. By Poincaré duality

$$H^2(\pi_1, \pi_2) \cong H_0(\pi_1, \pi_2) \cong \mathbf{Z}.$$

Therefore, $\mathcal{E}^2(X)$ is infinite cyclic. Marshall Cohen described to us a map representing a generator of $\mathcal{E}^2(X)$ as gotten by pinching a 2-sphere off T^2 at the base point and mapping it by a degree one map to the wedged S^2. It is interesting to note that this map induces a nontrivial automorphism of $H_2(X)$.

To complete the analysis of $\mathcal{E}(X)$, notice that the first k-invariant of X is zero since $\mathbf{Z} \oplus \mathbf{Z}$ is a 2-dimensional group. Therefore, $\operatorname{Aut}^k_{\pi_1}(\pi_2(X))$ is isomorphic to the group of units of $\mathbf{Z}(\pi_1(X))$. By Theorem 13 of Higman [**6**] the ring $\mathbf{Z}(\pi_1(X))$ has only trivial units. Thus $\mathcal{E}(X)$ is determined up to extension by the exact sequences

$$1 \to \mathcal{E}^1(X) \to \mathcal{E}(X) \to \mathrm{GL}_2(\mathbf{Z}) \to 1,$$
$$0 \to \mathbf{Z} \to \mathcal{E}^1(X) \to \pm \mathbf{Z} \oplus \mathbf{Z} \to 0.$$

In particular, $\mathcal{E}^2(X)$, $\mathcal{E}^1(X)$ and $\mathcal{E}(X)$ are all finitely generated.

EXAMPLE 4. Let $X = K \vee S^2$ where K is the finite 2-complex which models the presentation $(x, y;\ x^2 = y^3)$ of the trefoil knot group G. Then $\pi_1(X) \cong G$, $\pi_2(X) \cong \mathbf{Z}(G)$ and $\mathcal{E}^2(X) \cong H^2(G; \mathbf{Z}(G))$.

The center of G is generated by x^2; therefore, we have a central extension

$$0 \to \mathbf{Z} \to G \to Q \to 1$$

where $Q = \mathbf{Z}_2 * \mathbf{Z}_3$.

Next, feed this extension into the Lyndon-Hochschild-Serre spectral sequence to compute $H^2(G, \mathbf{Z}(G))$. One computes easily that $H^2(G, \mathbf{Z}(G)) \cong E_2^{1,1}$. By Poincaré duality we have

$$H^1(\mathbf{Z}, \mathbf{Z}(G)) \cong H_0(\mathbf{Z}, \mathbf{Z}(G)) \cong \mathbf{Z}(Q).$$

Therefore, $H^2(G, \mathbf{Z}(G)) \cong H^1(Q, \mathbf{Z}(Q))$. It is well known [**19**] that $H^1(Q, \mathbf{Z}(Q))$ is a free abelian group of countably infinite rank. Therefore, $\mathcal{E}^2(X)$ is a free abelian group of countably infinite rank.

The commutator subgroup of G is free of rank 2 and $G_{\text{ab}} \cong \mathbf{Z}$; therefore, G is indicable throughout by the lemma in the appendix of [**6**]. Hence, $\mathbf{Z}(G)$ has only trivial units. Thus, we have an exact sequence

$$0 \to \mathcal{E}^2(X) \to \mathcal{E}^1(X) \to \pm G \to 1.$$

At this point the reader should be wondering whether or not $\mathcal{E}^1(X)$ is infinitely generated. From the cochain complex obtained from the Lyndon resolution [**9**] of the one-relator group G, one sees that $H^2(G, \mathbf{Z}(G))$ is a cyclic right G-module; moreover, one sees from the description of $\mathcal{E}^1(X)$ in [**14**] that the action of $\pm G$ on $\mathcal{E}^2(X)$ induced by conjugation in $\mathcal{E}^1(X)$ corresponds to right translation in $H^2(G, \mathbf{Z}(G))$. This implies easily that $\mathcal{E}^1(X)$ is finitely generated.

Finally, we have an exact sequence

$$1 \to \mathcal{E}^1(X) \to \mathcal{E}(X) \to \operatorname{Aut} G \to 1.$$

The group Aut G is finitely generated [**16**]; therefore, $\mathcal{E}(X)$ is finitely generated.

Before we give our last example, which answers Question 3, we need to prove the following lemma.

LEMMA 1. *The group* $\mathrm{GL}_2(\mathbf{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}])$ *is infinitely generated if and only if*

$$\mathrm{SL}_2\big(\mathbf{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]\big)$$

is infinitely generated.

PROOF. Let $G = \mathrm{GL}_2(\mathbf{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}])$, $G_1 = \mathrm{SL}_2(\mathbf{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}])$, and $U = \mathbf{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]^*$. Then $U \cong \mathbf{Z}_2 \oplus \mathbf{Z}^n$ is generated multiplicatively by $-1, t_1, \ldots, t_n$. Because of the exact sequence

$$1 \to G_1 \to G \overset{\mathrm{Det}}{\to} U \to 1$$

the group G is infinitely generated only if G_1 is infinitely generated.

Conversely, suppose G_1 is infinitely generated. Let G_2 be the subgroup of G consisting of the matrices whose determinant is in the group generated by $t_1^2, \ldots, t_n^2$.

Then there is a retraction ρ: $G_2 \to G_1$ defined by

$$\rho(A) = \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} A \quad \text{where } \det A = u^2.$$

Therefore, G_2 is infinitely generated. But G_2 is a subgroup of index 2^{n+1} in G; therefore, G is infinitely generated. □

EXAMPLE 5. Let $X = T^2 \vee S^2 \vee S^2$. Then $\pi_1(X) \cong \mathbf{Z} \oplus \mathbf{Z}$ and $\pi_2(X) \cong \mathbf{Z}(\mathbf{Z} \oplus \mathbf{Z}) \oplus \mathbf{Z}(\mathbf{Z} \oplus \mathbf{Z})$. Therefore, we have an exact sequence

$$1 \to \mathcal{E}^1(X) \to \mathcal{E}(X) \to \mathrm{GL}_2(\mathbf{Z}) \to 1.$$

By Poincaré duality

$$H^2(\pi_1, \pi_2) \cong H_0(\pi_1, \pi_2) \cong \mathbf{Z} \oplus \mathbf{Z}.$$

Therefore, we have an exact sequence

$$0 \to \mathbf{Z} \oplus \mathbf{Z} \to \mathcal{E}^1(X) \to \mathrm{GL}_2(\mathbf{Z}(\mathbf{Z} \oplus \mathbf{Z})) \to 1.$$

The ring $\mathbf{Z}(\mathbf{Z} \oplus \mathbf{Z})$ is just $\mathbf{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}]$.

Recently, Bachmuth and Mochizuki [**2**] have shown that $\mathrm{SL}_2(\mathbf{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}])$ is infinitely generated. Therefore, by the lemma, $\mathrm{GL}_2(\mathbf{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}])$ is infinitely generated. This implies that $\mathcal{E}^1(X)$ is infinitely generated. However, it is unclear whether or not $\mathcal{E}(X)$ is finitely generated, since $\mathrm{GL}_2(\mathbf{Z})$ is finitely generated.

Question 5. Is $\mathcal{E}(T^2 \vee S^2 \vee S^2)$ infinitely generated?

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