FINITE 2-COMPLEXES WITH INFINITELY-GENERATED GROUPS OF SELF-HOMOTOPY-EQUIVALENCES

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Abstract. Examples of finite 2-dimensional aspherical cell complexes are given whose group of homotopy classes of self-homotopy-equivalences is infinitely generated.

In a recent paper [11], Darryl McCullough gave an example of a finite, 4-dimensional, aspherical cell complex whose group of homotopy classes of self-homotopy-equivalences is infinitely generated, and asked if there is a 2-dimensional example. In this paper, we give an infinite family of finite, 2-dimensional, aspherical cell complexes each of whose group of homotopy classes of self-homotopy-equivalences is infinitely generated.

McCullough's example is based on Lewin's [8] example of a finitely presented group with an infinitely generated automorphism group. Our examples are based on recent examples, given by the first named author [3], of one-relator groups with infinitely-generated automorphism group.

Let X be a path connected space with base point. We let $\mathcal{E}(X)$ denote the group of based homotopy classes of self-homotopy-equivalences, and $\mathcal{F}(X)$ the group of free homotopy classes of self-homotopy-equivalences. If X is an aspherical CW-complex with fundamental group $G$, then $\mathcal{E}(X) \cong \text{Aut } G$ and $\mathcal{F}(X) \cong \text{Out } G$ [20, pp. 225–226]. If $G$ is finitely generated, then $\text{Aut } G$ is finitely generated if and only if $\text{Out } G$ is finitely generated. Thus, if $X$ is a finite, aspherical complex, then $\mathcal{E}(X)$ is finitely generated if and only if $\mathcal{F}(X)$ is finitely generated.

Example 1. Let $n$ be a natural number greater than one, and consider the group

$$G(n) = \langle a, t; t^{-1}a^{-1}ta^n t^{-1}at = a \rangle.$$ 

The groups $\{G(n) | n = 2, 3, \ldots \}$ are mutually nonisomorphic, since $G(n)$ abelianizes to $\mathbb{Z}_{n-1} \oplus \mathbb{Z}$. In [3], it is shown that $\text{Out } G(n) \cong \mathbb{Z}[\frac{n}{2}]$—the additive group of rational numbers with denominator a nonnegative power of $n$. The group $\mathbb{Z}[\frac{n}{2}]$ is infinitely generated, since it is locally cyclic, but not cyclic. Thus $\text{Out } G(n)$ and $\text{Aut } G(n)$ are infinitely generated.

Let $X(n)$ be the 2-dimensional cell-complex which models the presentation of $G(n)$. Then $X(n)$ is a finite cell-complex (one 0-cell, two 1-cells and one 2-cell) with
fundamental group isomorphic to $G(n)$. As $X(n)$ has only one 2-cell, it follows from a lemma of Cockcroft [4] (see also [5]) that $X(n)$ is aspherical. Therefore, $\mathcal{G}(X(n)) \cong \text{Out}(G(n)) \cong \mathbb{Z}[\frac{1}{2}]$. Thus $\mathcal{G}(X(n))$ and $\tilde{\mathcal{G}}(X(n))$ are not finitely generated.

At this point we would like to thank the referee for asking the following interesting question.

**Question 1.** Is there a finite two-dimensional complex $X$ with $\text{Aut}(\pi_1(X))$ finitely-generated but $\tilde{\mathcal{G}}(X)$ not finitely-generated?

To see what answering this question would entail, let $\tilde{\mathcal{G}}(X)$ and $G(X)$ be the kernel and image respectively of the evaluation homomorphism

$$\tilde{\mathcal{G}}(X) \to \text{Aut}(\pi_1(X)).$$

Then we have an exact sequence

$$1 \to \tilde{\mathcal{G}}^1(X) \to \tilde{\mathcal{G}}(X) \to G(X) \to 1.$$

If $\tilde{\mathcal{G}}(X)$ is not finitely generated then either $G(X)$ or $\tilde{\mathcal{G}}^1(X)$ is not finitely generated. This suggests Questions 2 and 3.

**Question 2.** Is there a finite two-dimensional complex $X$ with $\text{Aut}(\pi_1(X))$ finitely-generated but $G(X)$ not finitely-generated?

This question appears to be difficult, since not much is known about the relation of $G(X)$ to $\text{Aut}(\pi_1(X))$. What is known is that if $X$ is aspherical or if $X$ has only one 2-cell (Jajodia [7]) then $G(X) = \text{Aut}(\pi_1(X))$, while Schellenberg [15] and Sieradski [18] have shown that $G(X)$ may be a proper subgroup of $\text{Aut}(\pi_1(X))$ when $\pi_1(X)$ is a noncyclic finite abelian group.

**Question 3.** Is there a finite 2-dimensional complex $X$ such that $\tilde{\mathcal{G}}^1(X)$ is not finitely generated?

Before we answer this question below, we need to analyze $\tilde{\mathcal{G}}^1(X)$ when $X$ is a finite 2-dimensional complex. Consider the evaluation homomorphism $\tilde{\mathcal{G}}^1(X) \to \text{Aut}_n(\pi_3(X))$. It follows from equation (1) in Adams [1] that the image is the subgroup $\text{Aut}_n(\pi_3(X))$ of all $\pi_1$-automorphisms of $\pi_3(X)$ which fix $k(X)$, the first $k$-invariant of $X$ in $H^3(\pi_1(X), \pi_3(X))$: moreover, Schellenberg [14] has shown that the kernel is isomorphic to $H^2(\pi_1(X), \pi_2(X))$. Thus we have an exact sequence

$$0 \to H^2(\pi_1(X), \pi_2(X)) \to \tilde{\mathcal{G}}^1(X) \to \text{Aut}_n(\pi_2(X)) \to 1.$$

In order to compute $\tilde{\mathcal{G}}^1(X)$, it would be convenient to have $k(X) = 0$, since then $\text{Aut}_n(\pi_3(X)) = \text{Aut}_n(\pi_3(X))$. This certainly is the case if the cohomological dimension of $\pi_1(X)$ is at most two, since $k(X)$ lives in a 3-dimensional cohomology group of $\pi_1(X)$. The following theorem says that, in fact, the converse is true.

**Theorem 1.** Let $k(X)$ be the first $k$-invariant of a 2-dimensional CW-complex $X$. Then $k(X) = 0$ if and only if the cohomological dimension of $\pi_1(X)$ is at most two.

**Proof.** Suppose $k(X) = 0$. Recall that $k(X)$ is represented by the crossed sequence [10]

$$0 \to \pi_2(X) \to \pi_2(\pi_1(X)) \to \pi_1(\pi_1(X)) \to \pi_1(X) \to 1$$

In order to compute $\tilde{\mathcal{G}}^1(X)$, it would be convenient to have $k(X) = 0$, since then $\text{Aut}_n(\pi_3(X)) = \text{Aut}_n(\pi_3(X))$. This certainly is the case if the cohomological dimension of $\pi_1(X)$ is at most two, since $k(X)$ lives in a 3-dimensional cohomology group of $\pi_1(X)$. The following theorem says that, in fact, the converse is true.
where $X^1$ is the 1-skeleton of $X$. Let $N = \text{Im} \partial_2$. Then $N$ is the group of relations of the presentation for $\pi_1(X)$ prescribed by the cell structure of $X$. By Theorem 8.1 of [12], the crossed extension

$$0 \to \pi_2(X) \to \pi_2(X, X^1) \to N \to 1$$

operator splits. As $\pi_2(X, X^1)$ is a free crossed module, we have that $N$ is a projective crossed module [13]. This implies that the relation module $N_{ab}$ is a projective $\pi_i(X)$-module. Therefore, $\text{cd} \pi_i(X) \leq 2$ by Serre [17, p. 90].

Next, we give an example which may possibly give an affirmative answer to Question 1.

**Example 2.** Let $X = S^1 \vee S^2 \vee S^2$. Then $\pi_1(X) \cong \mathbb{Z}$ and $\pi_2(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Clearly, $G(X) = \text{Aut}(\mathbb{Z}) = \pm 1$. As $\text{cd} \mathbb{Z} = 1$, we have that $H^i(\pi_i(X), \pi_2(X)) = 0$ for $i = 2, 3$; therefore, $\mathcal{E}_1(X) \cong \text{Aut}_n(\pi_2(X)) \cong \text{GL}_2(\mathbb{Z}(\mathbb{Z}))$. Thus we have an exact sequence

$$1 \to \text{GL}_2(\mathbb{Z}(\mathbb{Z})) \to \mathcal{E}_1(X) \to \pm 1 \to 1.$$ 

In particular, $\mathcal{E}(X)$ is finitely generated if and only if $\text{GL}_2(\mathbb{Z}(\mathbb{Z}))$ is finitely generated. Note that $\mathbb{Z}(\mathbb{Z})$ is just the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials in one variable $t$. Surprisingly, it is unknown whether or not $\text{GL}_2(\mathbb{Z}[t, t^{-1}])$ is finitely generated. Thus we ask

**Question 4.** Is the group $\text{GL}_2(\mathbb{Z}[t, t^{-1}])$ infinitely generated?

Note that an affirmative answer to Question 4 and Example 2 gives an affirmative answer to Question 1. Also, see Lemma 1 below.

Let $\mathcal{E}_2(X)$ be the kernel of the evaluation homomorphism

$$\mathcal{E}_1(X) \to \text{Aut}_n(\pi_2(X)).$$

Then $\mathcal{E}_2(X)$ is the group of homotopy classes of self-homotopy-equivalences of $X$ which induce the identity on both $\pi_1(X)$ and $\pi_2(X)$. Schellenberg remarked in [14] that he knew of no example of a finite 2-complex $X$ such that $\mathcal{E}_2(X)$ is nontrivial. Next, we give two such examples, one for which $\mathcal{E}_2(X)$ is finitely generated and the other for which $\mathcal{E}_2(X)$ is infinitely generated.

**Example 3.** Let $X = T^2 \vee S^2$ where $T^2$ is the torus. Then $\pi_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\pi_2(X) \cong \mathbb{Z}(\pi_1(X))$. By Poincaré duality

$$H^2(\pi_1, \pi_2) \cong H_0(\pi_1, \pi_2) \cong \mathbb{Z}.$$ 

Therefore, $\mathcal{E}_2(X)$ is infinite cyclic. Marshall Cohen described to us a map representing a generator of $\mathcal{E}_2(X)$ as gotten by pinching a 2-sphere off $T^2$ at the base point and mapping it by a degree one map to the wedged $S^2$. It is interesting to note that this map induces a nontrivial automorphism of $H_2(X)$.

To complete the analysis of $\mathcal{E}(X)$, notice that the first $k$-invariant of $X$ is zero since $\mathbb{Z} \oplus \mathbb{Z}$ is a 2-dimensional group. Therefore, $\text{Aut}^k_*(\pi_2(X))$ is isomorphic to the group of units of $\mathbb{Z}(\pi_1(X))$. By Theorem 13 of Higman [6] the ring $\mathbb{Z}(\pi_1(X))$ has only trivial units. Thus $\mathcal{E}(X)$ is determined up to extension by the exact sequences

$$1 \to \mathcal{E}_1(X) \to \mathcal{E}(X) \to \text{GL}_2(\mathbb{Z}) \to 1,$$

$$0 \to \mathbb{Z} \to \mathcal{E}_1(X) \to \pm \mathbb{Z} \oplus \mathbb{Z} \to 0.$$ 

In particular, $\mathcal{E}_2(X)$, $\mathcal{E}_1(X)$ and $\mathcal{E}(X)$ are all finitely generated.
Example 4. Let $X = K \vee S^2$ where $K$ is the finite 2-complex which models the presentation $(x, y; x^2 = y^3)$ of the trefoil knot group $G$. Then $\pi_1(X) \cong G$, $\pi_2(X) \cong \mathbb{Z}(G)$ and $\mathbb{E}^2(X) \cong H^2(G; \mathbb{Z}(G))$.

The center of $G$ is generated by $x^2$; therefore, we have a central extension

$$0 \to \mathbb{Z} \to G \to Q \to 1$$

where $Q = \mathbb{Z}_2 \ast \mathbb{Z}_3$.

Next, feed this extension into the Lyndon-Hochschild-Serre spectral sequence to compute $H^2(G, \mathbb{Z}(G))$. One computes easily that $H^2(G, \mathbb{Z}(G)) \cong E_2^{1,1}$. By Poincaré duality we have

$$H^1(\mathbb{Z}, \mathbb{Z}(G)) = H_0(\mathbb{Z}, \mathbb{Z}(G)) \cong \mathbb{Z}(Q).$$

Therefore, $H^2(G, \mathbb{Z}(G)) \cong H^1(Q, \mathbb{Z}(Q))$. It is well known [19] that $H^1(Q, \mathbb{Z}(Q))$ is a free abelian group of countably infinite rank. Therefore, $\mathbb{E}^2(X)$ is a free abelian group of countably infinite rank.

The commutator subgroup of $G$ is free of rank 2 and $G_{ab} \cong \mathbb{Z}$; therefore, $G$ is indicable throughout by the lemma in the appendix of [6]. Hence, $\mathbb{Z}(G)$ has only trivial units. Thus, we have an exact sequence

$$0 \to \mathbb{E}^1(X) \to \mathbb{E}^0(X) \to \pm G \to 1.$$ 

At this point the reader should be wondering whether or not $\mathbb{E}^0(X)$ is infinitely generated. From the cochain complex obtained from the Lyndon resolution [9] of the one-relator group $G$, one sees that $H^2(G, \mathbb{Z}(G))$ is a cyclic right $G$-module; moreover, one sees from the description of $\mathbb{E}^1(X)$ in [14] that the action of $\pm G$ on $\mathbb{E}^2(X)$ induced by conjugation in $\mathbb{E}^1(X)$ corresponds to right translation in $H^2(G, \mathbb{Z}(G))$. This implies easily that $\mathbb{E}^1(X)$ is finitely generated.

Finally, we have an exact sequence

$$1 \to \mathbb{E}^1(X) \to \mathbb{E}(X) \to \text{Aut} G \to 1.$$ 

The group $\text{Aut} G$ is finitely generated [16]; therefore, $\mathbb{E}(X)$ is finitely generated.

Before we give our last example, which answers Question 3, we need to prove the following lemma.

**Lemma 1.** The group $GL_2(\mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}])$ is infinitely generated if and only if $SL_2(\mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}])$ is infinitely generated.

**Proof.** Let $G = GL_2(\mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}])$, $G_1 = SL_2(\mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}])$, and $U = \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. Then $U \cong \mathbb{Z}_2 \oplus \mathbb{Z}^n$ is generated multiplicatively by $-1, t_1, \ldots, t_n$. Because of the exact sequence

$$1 \to G_1 \to G \to U \to 1,$$

the group $G$ is infinitely generated only if $G_1$ is infinitely generated.

Conversely, suppose $G_1$ is infinitely generated. Let $G_2$ be the subgroup of $G$ consisting of the matrices whose determinant is in the group generated by $t_1^2, \ldots, t_n^2$. 


Then there is a retraction $\rho: G_2 \to G_1$ defined by

$$\rho(A) = \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} A \quad \text{where } \det A = u^2.$$  

Therefore, $G_2$ is infinitely generated. But $G_2$ is a subgroup of index $2^{n+1}$ in $G$; therefore, $G$ is infinitely generated. \[
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**Example 5.** Let $X = T^2 \vee S^2 \vee S^2$. Then $\pi_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\pi_2(X) \cong \mathbb{Z}\langle\mathbb{Z} \oplus \mathbb{Z}\rangle \oplus \mathbb{Z}\langle\mathbb{Z} \oplus \mathbb{Z}\rangle$. Therefore, we have an exact sequence

$$1 \to \mathcal{C}(X) \to \mathcal{E}(X) \to \text{GL}_2(\mathbb{Z}) \to 1.$$  

By Poincaré duality

$$H^2(\pi_1, \pi_2) \cong H_0(\pi_1, \pi_2) \cong \mathbb{Z} \oplus \mathbb{Z}.$$  

Therefore, we have an exact sequence

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \mathcal{C}(X) \to \text{GL}_2(\mathbb{Z}\langle\mathbb{Z} \oplus \mathbb{Z}\rangle) \to 1.$$  

The ring $\mathbb{Z}\langle\mathbb{Z} \oplus \mathbb{Z}\rangle$ is just $\mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}]$.

Recently, Bachmuth and Mochizuki [2] have shown that $\text{SL}_2(\mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}])$ is infinitely generated. Therefore, by the lemma, $\text{GL}_2(\mathbb{Z}\langle t_1, t_1^{-1}, t_2, t_2^{-1} \rangle)$ is infinitely generated. This implies that $\mathcal{C}(X)$ is infinitely generated. However, it is unclear whether or not $\mathcal{C}(X)$ is finitely generated, since $\text{GL}_2(\mathbb{Z})$ is finitely generated.

**Question 5.** Is $\mathcal{C}(T^2 \vee S^2 \vee S^2)$ infinitely generated?

**References**

2. S. Bachmuth and H. Y. Mochizuki, $E_3 \neq \text{SL}_2$ for most Laurent polynomial rings, Preprint, Univ. of California at Santa Barbara, 1981.


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