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PERFECTLY NORMAL COMPACT SPACES ARE CONTINUOUS IMAGES OF $\beta N \setminus N$

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ABSTRACT. Every perfectly normal compact space is a continuous image of $N^* = \beta N \setminus N$.

It is well known that a space X is a continuous image of $N^* = \beta N \setminus N$ if and only if it is a remainder of some compactification aN of natural numbers and that every continuous image of N^* is a compact space of weight $\leqslant c = 2^\omega$. It is also easy to show that every separable compact space is a continuous image of N^*.

Parovičenko proved in 1963 [**P**] that every compact space of weight $\leqslant \omega_1$ is a continuous image of N^* (see [**BS**] for a nice proof of this theorem). Thus under CH a compact space is a continuous image of N^* if and only if its weight is $\leqslant c$. On the other hand, it is consistent with the axioms of ZFC that $\omega_2 \leqslant c$ and that the space $\omega_2 + 1$ of ordinals $\leqslant \omega_2$ is a continuous image of N^* (see e.g. [**FG**]) and also that $\omega_2 \leqslant c$ and every compact space of weight $\leqslant c$ is a continuous image of N^* (J. Baumgartner). Martin's Axiom implies that every compact space of weight $< c$ is a continuous image of N^* [**vDP**], whereas it is also consistent with Martin's Axiom that there exists a compact space of weight c which is not a continuous image of N^* (J. Baumgartner, R. Frankiewicz).

The following result however does not depend on additional axioms of set theory.

THEOREM. *Every perfectly normal compact space is a continuous image of N^*.*

The above theorem does not follow from the results mentioned earlier since it is consistent with the axioms of set theory that there exist perfectly normal compact spaces which are neither separable nor of weight $\leqslant \omega_1$. The proof of the theorem is based on the technique employed by A. Błaszczyk and A. Szymański in [**BS**]. We also use the result, due to B. Šapirovski [**Š**], asserting that the π-weight of a perfectly normal compact space is $\leqslant \omega_1$. The example below shows that this latter property in itself is insufficient.

EXAMPLE. It is consistent with the axioms of ZFC that there exists a compact space of weight $\omega_2 \leqslant c$ and π-weight ω_1 which is not a continuous image of N^*.

Let us recall that a compact space is *perfectly normal* if and only if all of its closed subsets are G_δ's. Every perfectly normal compact space is first countable, (hereditarily) ccc and has weight $\leqslant c$. The following two problems arise in a natural way:

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Problem 1 [**vDP**]. Is every first countable compact space a continuous image of N^*?

Problem 2. Is every (hereditarily) ccc compact space of weight $\leq c$ a continuous image of N^*?

Let us also recall that a family $\mathfrak{B}$ of nonempty open subsets of X is a *π-base* if every nonempty open subset of X contains a member of $\mathfrak{B}$. The *π-weight* of a space X is the minimal cardinality of its π-bases. A continuous mapping $f\colon X \to Y$ of X onto Y is *irreducible* if there is no closed proper subset F of X with $f(F) = Y$. (The author wishes to express his thanks to Charles Mills for observing that a more complicated property originally used in the formulation of the lemma below is equivalent to irreduciblity.) By I we denote the unit interval and if $\varphi\colon n \to 2$ is a function of n into 2, where $n = \{0, 1, \ldots, n-1\}$ is a natural number and if $i \in 2$, then by $\varphi^\frown \langle i \rangle$ we denote a function $\psi\colon n+1 \to 2$ defined as follows: $\psi(j) = \varphi(j)$ for $j < n$ and $\psi(n) = i$.

LIFTING LEMMA. *Let X be compact and perfectly normal, Z a closed subspace of $X \times I$ and suppose that the restriction $\pi \restriction Z\colon Z \to X$ of the projection $\pi\colon X \times I \to X$ is irreducible.*[2]

If $f\colon N^* \to X$ is a continuous mapping of N^* onto X, then there exists a continuous mapping $g\colon N^* \to Z$ of N^* onto Z such that $f = \pi \circ g$.

PROOF. For every sequence $\varphi \in 2^{<\omega}$ define a subinterval I_φ of I as follows:
$I_\varnothing = I$;
$I_{\langle 0 \rangle} = [0, \frac{1}{2}]$; $I_{\langle 1 \rangle} = [\frac{1}{2}, 1]$;
$I_{\langle 0,0 \rangle} = [0, \frac{1}{4}]$; $I_{\langle 0,1 \rangle} = [\frac{1}{4}, \frac{1}{2}]$; $I_{\langle 1,0 \rangle} = [\frac{1}{2}, \frac{3}{4}]$; etc. ....

For every $\varphi \in 2^{<\omega}$ define $F_\varphi = \pi(Z \cap (X \times I_\varphi))$. Clearly F_φ is a closed subset of X. For every φ by recursion we construct a clopen subset C_φ of N^* so that the following conditions are satisfied:

(1) $C_\varnothing = N^*$;
(2) $C_\varphi = C_{\varphi^\frown\langle 0 \rangle} \cup C_{\varphi^\frown\langle 1 \rangle}$;
(3) $C_{\varphi^\frown\langle 0 \rangle} \cap C_{\varphi^\frown\langle 1 \rangle} = \varnothing$;
(4) $f(C_\varphi) \subset F_\varphi(1)$.

Suppose that C_φ has been constructed. The sets

$$U_0 = f^{-1}\left(F_\varphi \setminus F_{\varphi^\frown\langle 1 \rangle}\right) \cap C_\varphi$$

and

$$U_1 = f^{-1}\left(F_\varphi \setminus F_{\varphi^\frown\langle 0 \rangle}\right) \cap C_\varphi$$

are disjoint, open, F_σ-subsets of C_φ. Since N^* is an F-space (cf. [**GJ**]) there exists a clopen set $W \subset C_\varphi$ such that

$$W \supset U_0 \quad \text{and} \quad C_\varphi \setminus W \supset U_1.$$

[2] If in the Lifting Lemma one additionally assumes that X is hereditarily separable (e.g. metrizable) then the assumption that $\pi \restriction Z$ is irreducible can be dropped, (4) can be replaced by "$f(C_\varphi) = F_\varphi$" and the Lemma so modified yields a proof of Parovičenko's theorem (cf. [**BS**]).

Put $C_{\varphi^\frown\langle 0\rangle} = W$ and $C_{\varphi^\frown\langle 1\rangle} = C_\varphi \setminus W$. Since $C_{\varphi^\frown\langle 0\rangle} \cap f^{-1}(F_\varphi \setminus F_{\varphi^\frown\langle 0\rangle}) = \varnothing$ and $f(C_{\varphi^\frown\langle 0\rangle}) \subset F_\varphi$, we have $f(C_{\varphi^\frown\langle 0\rangle}) \subset F_{\varphi^\frown\langle 0\rangle}$ and similarly $f(C_{\varphi^\frown\langle 1\rangle}) \subset F_{\varphi^\frown\langle 1\rangle}$.

Define a mapping $h: N^* \to I$ as follows. For every $y \in N^*$ there exists exactly one $\Phi(y) \in 2^\omega$ such that $y \in \bigcap_{n<\omega} C_{\Phi(y)\restriction n}$. Define $h(y)$ so that $\{h(y)\} = \bigcap_{n<\omega} I_{\Phi(y)\restriction n}$. One easily checks that h is continuous.

Define $g: N^* \to X \times I$ by $g(y) = (f(y), h(y))$, for $y \in N^*$. Clearly g is continuous and for every $y \in N^*$ we have $\pi(g(y)) = f(y)$, thus it suffices to prove that $g(N^*) = Z$. We first show that $g(N^*) \subset Z$.

Suppose otherwise. Then there exists a $y \in N^*$ such that $(f(y), h(y)) \notin Z$. Fix open sets U in X and V in I such that $f(y) \in U$ and $h(y) \in V$ and $(U \times V) \cap Z = \varnothing$. There exists an $n < \omega$ such that $V \supset I_{\Phi(y)\restriction n}$. Therefore $U \cap F_{\Phi(y)\restriction n} = \varnothing$, which is impossible because $f(y) \in U \cap f(C_{\Phi(y)\restriction n}) \subset U \cap F_{\Phi(y)\restriction n}$.

Let $F = g(N^*) \subset Z$. Then $\pi(F) = \pi(g(N^*)) = f(N^*) = X$ and thus by the irreducibility of $\pi \restriction Z$ we have $F = Z$. $\square$

Proof of the theorem. Let X be a perfectly normal compact space of weight κ. By Šapirovski's theorem **[Š]**, X has a π-base of cardinality $\leqslant \omega_1$. Let $\{B_\alpha: \alpha < \kappa\}$ be a base of X such that the family $\{B_\alpha: \alpha < \omega_1\}$ forms a π-base of X. For every $\alpha < \kappa$ let $h_\alpha: X \to I_\alpha = I$ be continuous and such that $h_\alpha^{-1}((0,1]) = B_\alpha$. The diagonal mapping $h = \triangle_{\alpha<\kappa} h_\alpha: X \to \prod_{\alpha<\kappa} I_\alpha$ is a homeomorphic embedding (cf. **[E]**) and thus we can identify X with $h(X)$. For $\alpha \leqslant \kappa$ let $I^\alpha = \prod_{\beta<\alpha} I_\beta$ and for $\alpha < \beta \leqslant \kappa$ let $\pi_\alpha^\beta: I^\beta \to I^\alpha$ be the projection and let $\pi_\alpha = \pi_\alpha^\kappa: I^\kappa \to I^\alpha$. Put $X_\alpha = \pi_\alpha(X)$. Then $X = \varprojlim\{X_\alpha, \pi_\alpha^\beta, \alpha < \beta < \kappa\}$, i.e. X is the inverse limit of X_α's. Thus by **[E**, Corollary 3.2.16] to prove the theorem it suffices to construct for every $\alpha < \kappa$ a continuous mapping $f_\alpha: N^* \to X_\alpha$ of N^* onto X_α so that

$$(*) \qquad f_\alpha = \pi_\alpha^\beta \circ f_\beta,$$

for every $\alpha < \beta < \kappa$.

Let $f_{\omega_1}: N^* \to X_{\omega_1}$ be an arbitrary continuous mapping of N^* onto X_{ω_1} existing in virtue of Parovičenko's theorem **[P]** and for all $\beta < \omega_1$ put $f_\beta = \pi_\beta^{\omega_1} \circ f_{\omega_1}: N^* \to X_\beta$. Now suppose that $\omega_1 < \alpha < \kappa$ and that for all $\beta < \alpha$ we have already constructed mappings f_β satisfying $(*)$. If α is a limit ordinal, then—again using **[E**, Corollary 3.2.16]—we put

$$f_\alpha = \varprojlim \{f_\beta: \beta < \alpha\}: N^* \to X_\alpha.$$

Otherwise, $\alpha = \beta + 1$ for some $\beta \geqslant \omega_1$ and the mapping $f_\beta: N^* \to X_\beta$ is defined. Clearly $X_\alpha \subset X_\beta \times I_\beta$ and $\pi_\beta^\alpha \restriction X_\alpha: X_\alpha \to X_\beta$. By the Lifting Lemma, it suffices to prove that $\pi_\beta^\alpha \restriction X_\alpha$ is irreducible. Suppose that F is a closed subspace of X_α, $\pi_\beta^\alpha \restriction F$ maps F onto X_β, $U = X_\alpha \setminus F \neq \varnothing$ and let $G = \pi_\alpha^{-1}(U) \cap X$. There exists a $\gamma < \omega_1 < \alpha$ such that $\varnothing \neq B_\gamma \subset G$, i.e. $\varnothing \neq p_\gamma^{-1}((0,1]) \cap X \subset G$, where $p_\gamma: I^\kappa \to I_\gamma$ is the projection. Choose $x \in p_\gamma^{-1}((0,1]) \cap X$. Then $y = \pi_\beta(x) \in X_\beta$ and $\pi_\beta^{-1}(y) \cap X \subset G = \pi_\alpha^{-1}(U) \cap X$, hence $(\pi_\beta^\alpha)^{-1}(y) \cap X_\alpha \subset U$. Therefore $y \notin \pi_\beta^\alpha(F)$, a contradiction. $\square$

Construction of the example. It is consistent with the axioms of ZFC that there exists a completely regular first countable space Z of cardinality ω_2 and density

ω_1 which cannot be embedded into any regular first countable separable space (see **[vDP]**). Let X be an arbitrary compactification of Z of weight ω_2. Since Z is dense in X, points of Z have countable character in X. Thus X has a π-base of cardinality ω_1, because it contains a dense set of cardinality ω_1 consisting of points of countable character. Suppose that $X = aN \setminus N$ for some compactification aN of N. Then $Y = Z \cup N \subset aN$ would be a first countable and separable extension of Z, which is impossible. □

ADDED IN PROOF. M. Bell constructed a consistent example of a ccc compact space of weight c which is not a continuous image of N^*.

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