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ON SIMPLE LOOPS ON A SOLID TORUS OF GENERAL GENUS

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ABSTRACT. Let l be a simple loop on the boundary of a solid torus T of genus g. Let $\mathbf{m}$ be a complete system of oriented meridian disks of T. Let $W(l, \mathbf{m})$ be a word obtained by reading the intersections $l \cap \mathbf{m}$ along l. We shall give a natural method for realizing the cyclic reductions of $W(l, \mathbf{m})$ geometrically. This yields a simple proof of Whitehead-Zieschang's theorem related to the minimality of the intersections of two systems $\{l_1, \ldots, l_n\}$ and $\mathbf{m}$.

1. Statement of results. Let T be a solid torus of genus g, i.e., a 3-ball with g 1-handles, and ∂T be the boundary of T. Let $\mathbf{m}$ be a complete system $\{m_1, \ldots, m_g\}$ of oriented meridian disks of T (here, "system" means "mutually disjoint set" and "complete" means that T being cut open by $\mathbf{m}$ is a 3-ball). A simple loop l on ∂T is said to be *normal* to $\mathbf{m}$ if any intersection of $l \cap \mathbf{m}$ $(= l \cap (m_1 \cup \cdots \cup m_g))$ is transversal and cannot be removed by an isotopy on ∂T. Let $W(l, \mathbf{m})$ be a (cyclic) word in the symbols $a_1, \ldots, a_g$ given by reading the intersections $l \cap \mathbf{m}$ along l, where $l \cap m_i$ corresponds to the letter a_i, $i = 1, \ldots, g$. A disk m in T is said to be *proper* if $m \cap \partial T = \partial m$.

THEOREM 1. *Let l be a simple loop on ∂T which is normal to $\mathbf{m}$. If a word $W(l, \mathbf{m})$ is not cyclically reduced, there is an algorithm to find a system $\hat{\mathbf{m}}$ of oriented proper disks $\hat{m}_1, \ldots, \hat{m}_k$ $(k \geqslant g)$ of T such that*

(1) *each disk $\hat{m}_i$ is labeled by one of the symbols $a_1, \ldots, a_g$,*

(2) *$\hat{\mathbf{m}}$ contains a complete subsystem $\mathbf{m}_0 = \{\hat{m}_{i_1}, \ldots, \hat{m}_{i_g}\}$ of meridian disks where the label of $\hat{m}_{i_j}$ is the letter a_j, $j = 1, \ldots, g$,*

(3) *a word $W(l, \hat{\mathbf{m}})$ obtained by reading the intersections $l \cap \hat{\mathbf{m}}$ along l coincides with the cyclically reduced word $\tilde{W}(l, \mathbf{m})$ obtained from $W(l, \mathbf{m})$.*

Using Whitehead's result [5], Zieschang [6] showed the following theorem, which implies the existence of the algorithm to transform any Heegaard diagram to the pseudominimal one (cf. [2]). Theorem 1 simplifies Zieschang's proof of it in a natural way.

THEOREM 2 (WHITEHEAD-ZIESCHANG). *For any system $\{l_1, \ldots, l_n\}$ of simple loops on ∂T, there is an algorithm to find a complete system $\mathbf{m}$ of meridian disks of T which has minimal intersections with $\{l_1, \ldots, l_n\}$.*

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2. Proof of Theorem 1. Let α be a subarc of l corresponding to a cancellable part $a_i a_i^{-1}$ or $a_i^{-1} a_i$ of $W(l, \mathbf{m})$ in the cyclic sense. By the surgery on ∂m_i along α, we have two oriented simple loops which bound oriented proper disks m_i', m_i'' in T, respectively. Replacing m_i by m_i' and m_i'', we have a new system $\mathbf{m}_1$ of oriented proper disks of T from $\mathbf{m}$. Labeling m_i' and m_i'' by the same letter a_i, it is easy to see that $\mathbf{m}_1$ satisfies (1) and (2). If $W(l, \mathbf{m}_1)$ is not cyclically reduced, we can repeat this process and get a new system $\mathbf{m}_2$ satisfying (1) and (2) though, in this case, the surgery may coincide with "a band move" (cf. [**1**]). (Note that a band move decreases the number of disks of the system by exactly 1.) Repeating this process until $W(l, \mathbf{m}_p)$ is cyclically reduced, we have the desired system $\hat{\mathbf{m}} = \mathbf{m}_p$ which satisfies (1), (2) and (3).

REMARK 1. Theorem 1 is also valid if $\mathbf{m}$ is a system of oriented proper disks $m_1', \ldots, m_n'$ $(n \geqslant g)$ of T such that

(i) each disk m_i' is labeled by one of the symbols $a_1, \ldots, a_g$,

(ii) let Λ_j $(j = 1, \ldots, g)$ be a subindex-set $\{i \mid m_i' \text{ is labeled by the letter } a_j\}$ then the subset $\{\Sigma_{i \in \Lambda_1}[\partial m_i'], \ldots, \Sigma_{i \in \Lambda_g}[\partial m_i']\}$ of the homology group $H_1(\partial T)$ is linearly independent in $H_1(\partial T)$, where [ ] means a homology class.

REMARK 2. In the case of genus 2, we can get a sharper result in [**3**] where the above surgerical method was used first.

3. Proof of Theorem 2. By Whitehead's algorithm [**5**] (also, see [**4, 6**]), we can find a complete system $\mathbf{m}'$ of (oriented) meridian disks of T such that

(a) $\{l_1, \ldots, l_n\}$ is normal to $\mathbf{m}'$,

(b) the length $\Sigma_{i=1}^n L(\tilde{W}(l_i, \mathbf{m}'))$ of the set of cyclically reduced words $\tilde{W}(l_1, \mathbf{m}'), \ldots, \tilde{W}(l_n, \mathbf{m}')$ is minimal, i.e., $\Sigma_{i=1}^n L(\tilde{W}(l_i, \mathbf{m}')) \leqslant \Sigma_{i=1}^n L(\tilde{W}(l_i, \mathbf{m}''))$ for any complete system $\mathbf{m}''$.

Applying Theorem 1 to l_1 and $\mathbf{m}'$, we can find a system $\hat{\mathbf{m}}_1$ satisfying (1), (2), $\tilde{W}(l_1, \mathbf{m}') = W(l_1, \hat{\mathbf{m}}_1)$ and $W(l_i, \mathbf{m}') = W(l_i, \hat{\mathbf{m}}_1)$, $i = 2, \ldots, n$. By Remark 1, we can apply Theorem 1 to l_2 and $\hat{\mathbf{m}}_1$, and find a system $\hat{\mathbf{m}}_2$ satisfying (1), (2), $\hat{W}(l_1, \mathbf{m}') = W(l_1, \hat{\mathbf{m}}_2)$, $\tilde{W}(l_2, \mathbf{m}') = W(l_2, \hat{\mathbf{m}}_2)$ and $W(l_i, \mathbf{m}') = W(l_i, \hat{\mathbf{m}}_2)$, $i = 3, \ldots, n$. Repeating this process n-times, we can find a system $\hat{\mathbf{m}}_n$ satisfying (1), (2) and $\tilde{W}(l_i, \mathbf{m}') = W(l_i, \hat{\mathbf{m}}_n)$, $i = 1, \ldots, n$. Let $\mathbf{m}_0$ be a subsystem of $\hat{\mathbf{m}}_n$ by (2). It holds that

$$\sum_{i=1}^n L(\tilde{W}(l_i, \mathbf{m}')) \left(= \sum_{i=1}^n L(W(l_i, \hat{\mathbf{m}}_n)) \right) \geqslant \sum_{i=1}^n L(W(l_i, \mathbf{m}_0)).$$

Thus we can find the desired $\mathbf{m} = \mathbf{m}_0$.

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