

ON SIMPLE LOOPS ON A SOLID TORUS OF GENERAL GENUS

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ABSTRACT. Let l be a simple loop on the boundary of a solid torus T of genus g . Let \mathbf{m} be a complete system of oriented meridian disks of T . Let $W(l, \mathbf{m})$ be a word obtained by reading the intersections $l \cap \mathbf{m}$ along l . We shall give a natural method for realizing the cyclic reductions of $W(l, \mathbf{m})$ geometrically. This yields a simple proof of Whitehead-Zieschang's theorem related to the minimality of the intersections of two systems $\{l_1, \dots, l_n\}$ and \mathbf{m} .

1. Statement of results. Let T be a solid torus of genus g , i.e., a 3-ball with g 1-handles, and ∂T be the boundary of T . Let \mathbf{m} be a complete system $\{m_1, \dots, m_g\}$ of oriented meridian disks of T (here, "system" means "mutually disjoint set" and "complete" means that T being cut open by \mathbf{m} is a 3-ball). A simple loop l on ∂T is said to be *normal* to \mathbf{m} if any intersection of $l \cap \mathbf{m}$ ($= l \cap (m_1 \cup \dots \cup m_g)$) is transversal and cannot be removed by an isotopy on ∂T . Let $W(l, \mathbf{m})$ be a (cyclic) word in the symbols a_1, \dots, a_g given by reading the intersections $l \cap \mathbf{m}$ along l , where $l \cap m_i$ corresponds to the letter a_i , $i = 1, \dots, g$. A disk m in T is said to be *proper* if $m \cap \partial T = \partial m$.

THEOREM 1. *Let l be a simple loop on ∂T which is normal to \mathbf{m} . If a word $W(l, \mathbf{m})$ is not cyclically reduced, there is an algorithm to find a system $\hat{\mathbf{m}}$ of oriented proper disks $\hat{m}_1, \dots, \hat{m}_k$ ($k \geq g$) of T such that*

- (1) *each disk \hat{m}_i is labeled by one of the symbols a_1, \dots, a_g ,*
- (2) *$\hat{\mathbf{m}}$ contains a complete subsystem $\mathbf{m}_0 = \{\hat{m}_1, \dots, \hat{m}_g\}$ of meridian disks where the label of \hat{m}_j is the letter a_j , $j = 1, \dots, g$,*
- (3) *a word $\tilde{W}(l, \hat{\mathbf{m}})$ obtained by reading the intersections $l \cap \hat{\mathbf{m}}$ along l coincides with the cyclically reduced word $\tilde{W}(l, \mathbf{m})$ obtained from $W(l, \mathbf{m})$.*

Using Whitehead's result [5], Zieschang [6] showed the following theorem, which implies the existence of the algorithm to transform any Heegaard diagram to the pseudominimal one (cf. [2]). Theorem 1 simplifies Zieschang's proof of it in a natural way.

THEOREM 2 (WHITEHEAD-ZIESCHANG). *For any system $\{l_1, \dots, l_n\}$ of simple loops on ∂T , there is an algorithm to find a complete system \mathbf{m} of meridian disks of T which has minimal intersections with $\{l_1, \dots, l_n\}$.*

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2. Proof of Theorem 1. Let α be a subarc of l corresponding to a cancellable part $a_i a_i^{-1}$ or $a_i^{-1} a_i$ of $W(l, \mathbf{m})$ in the cyclic sense. By the surgery on ∂m_i along α , we have two oriented simple loops which bound oriented proper disks m'_i, m''_i in T , respectively. Replacing m_i by m'_i and m''_i , we have a new system \mathbf{m}_1 of oriented proper disks of T from \mathbf{m} . Labeling m'_i and m''_i by the same letter a_i , it is easy to see that \mathbf{m}_1 satisfies (1) and (2). If $W(l, \mathbf{m}_1)$ is not cyclically reduced, we can repeat this process and get a new system \mathbf{m}_2 satisfying (1) and (2) though, in this case, the surgery may coincide with "a band move" (cf. [1]). (Note that a band move decreases the number of disks of the system by exactly 1.) Repeating this process until $W(l, \mathbf{m}_p)$ is cyclically reduced, we have the desired system $\hat{\mathbf{m}} = \mathbf{m}_p$ which satisfies (1), (2) and (3).

REMARK 1. Theorem 1 is also valid if \mathbf{m} is a system of oriented proper disks m'_1, \dots, m'_n ($n \geq g$) of T such that

- (i) each disk m'_i is labeled by one of the symbols a_1, \dots, a_g ,
- (ii) let Λ_j ($j = 1, \dots, g$) be a subindex-set $\{i \mid m'_i \text{ is labeled by the letter } a_j\}$ then the subset $\{\sum_{i \in \Lambda_1} [\partial m'_i], \dots, \sum_{i \in \Lambda_g} [\partial m'_i]\}$ of the homology group $H_1(\partial T)$ is linearly independent in $H_1(\partial T)$, where $[\]$ means a homology class.

REMARK 2. In the case of genus 2, we can get a sharper result in [3] where the above surgical method was used first.

3. Proof of Theorem 2. By Whitehead's algorithm [5] (also, see [4, 6]), we can find a complete system \mathbf{m}' of (oriented) meridian disks of T such that

- (a) $\{l_1, \dots, l_n\}$ is normal to \mathbf{m}' ,
- (b) the length $\sum_{i=1}^n L(\tilde{W}(l_i, \mathbf{m}'))$ of the set of cyclically reduced words $\tilde{W}(l_1, \mathbf{m}'), \dots, \tilde{W}(l_n, \mathbf{m}')$ is minimal, i.e., $\sum_{i=1}^n L(\tilde{W}(l_i, \mathbf{m}')) \leq \sum_{i=1}^n L(\tilde{W}(l_i, \mathbf{m}''))$ for any complete system \mathbf{m}'' .

Applying Theorem 1 to l_1 and \mathbf{m}' , we can find a system $\hat{\mathbf{m}}_1$ satisfying (1), (2), $\tilde{W}(l_1, \mathbf{m}') = W(l_1, \hat{\mathbf{m}}_1)$ and $W(l_i, \mathbf{m}') = W(l_i, \hat{\mathbf{m}}_1)$, $i = 2, \dots, n$. By Remark 1, we can apply Theorem 1 to l_2 and $\hat{\mathbf{m}}_1$, and find a system $\hat{\mathbf{m}}_2$ satisfying (1), (2), $\tilde{W}(l_1, \mathbf{m}') = W(l_1, \hat{\mathbf{m}}_2)$, $\tilde{W}(l_2, \mathbf{m}') = W(l_2, \hat{\mathbf{m}}_2)$ and $W(l_i, \mathbf{m}') = W(l_i, \hat{\mathbf{m}}_2)$, $i = 3, \dots, n$. Repeating this process n -times, we can find a system $\hat{\mathbf{m}}_n$ satisfying (1), (2) and $\tilde{W}(l_i, \mathbf{m}') = W(l_i, \hat{\mathbf{m}}_n)$, $i = 1, \dots, n$. Let \mathbf{m}_0 be a subsystem of $\hat{\mathbf{m}}_n$ by (2). It holds that

$$\sum_{i=1}^n L(\tilde{W}(l_i, \mathbf{m}')) \left(= \sum_{i=1}^n L(W(l_i, \hat{\mathbf{m}}_n)) \right) \geq \sum_{i=1}^n L(W(l_i, \mathbf{m}_0)).$$

Thus we can find the desired $\mathbf{m} = \mathbf{m}_0$.

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