A GENERALIZATION OF A THEOREM OF AYOUB AND CHOWLA

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ABSTRACT. Let \( \chi_1 \) and \( \chi_2 \) be characters modulo \( q_1 \) and \( q_2 \), respectively, where \( q_1 \) and \( q_2 \) are positive integers. Let

\[
 f(n) = \sum_{d|n} \chi_1(d)\chi_2(n/d).
\]

In this paper we shall give an estimate for the sum

\[
 \sum_{n \leq x} f(n) \log(x/n).
\]

1. Introduction. Let \( \chi_1 \) and \( \chi_2 \) be characters modulo \( q_1 \) and \( q_2 \), respectively, where \( q_1 \) and \( q_2 \) are positive integers. Let

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\]

In [2] Ayoub and Chowla considered the case \( \chi_1 \equiv 1 \), \( q_1 = 1 \) and \( \chi_2 \) the Kronecker character. In [3] Müller considered the case \( \chi_1 \equiv 1 \) and \( \chi_2 \) the nonprincipal character modulo 4.

We shall prove the following theorem.

THEOREM. We have, as \( x \to \infty \),

\[
 \sum_{n \leq x} f(n) \log(x/n) = C_1(\chi_1, \chi_2)x \log x + C_2(\chi_1, \chi_2)x
\]

\[
 + C_3(\chi_1, \chi_2)\log x + C_4(\chi_1, \chi_2) + O(x^{-2/3}),
\]

where the \( C_j(\chi_1, \chi_2), 1 \leq j \leq 4 \), are certain constants to be determined below (see (3.10) and (4.5)).

In what follows we shall use the notation \( \int_{a-iT}^{a+iT} \) to stand for the integral \( \int_{a-iT}^{a+iT} \).

2. Lemmas.

LEMMA 1. Let \( \chi \) be a character modulo \( q \) and let \( L(s, \chi) \) be the associated Dirichlet \( L \) series. Then, for \( s \neq 1 \), we have

\[
 L(s, \chi) = q^{-s} \sum_{n=1}^{q} \chi(n) \zeta(s, n/q),
\]

where \( \zeta(s, n/q) \) is the Hurwitz zeta function.

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PROOF. For \( \text{Re}(s) > 1 \), we have

\[
L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s} = \sum_{m=0}^{q} \sum_{n=1}^{\infty} \chi(mq + n)(mq + n)^{-s} = \sum_{n=1}^{q} \chi(n) \sum_{m=0}^{\infty} (mq + n)^{-s} = q^{-s} \sum_{n=1}^{q} \chi(n) \zeta(s, n/q).
\]

Now \( \zeta(s, n/q) \) can be continued to a meromorphic function in the entire complex plane whose only singularity is a simple pole at \( s = 1 \) (see [5, §13.13]). Since \( L(s, \chi) \) can be continued to a meromorphic function in the complex plane with at most a simple pole at \( s = 1 \) the result follows by analytic continuation and completes the proof of the lemma.

**Lemma 2.** (1) Let \( 0 < a < 1 \). Then we have, as \( |t| \rightarrow \infty \),

\[
\zeta(a + it, a) \sim \begin{cases} 
1 & \text{if } a > 1, \\
|t|^{(1-\sigma)/2} \log |t| & \text{if } 0 \leq \sigma < 1, \\
|t|^{1/2 - \sigma} \log |t| & \text{if } \sigma < 0.
\end{cases}
\]

(2) Let \( L(s, \chi) \) be as in Lemma 1. Then we have, as \( |t| \rightarrow \infty \),

\[
L(a + it, \chi) \sim \begin{cases} 
1 & \text{if } a > 1, \\
(q|t|)^{(1-\sigma)/2} \log(q|t|) & \text{if } 0 \leq \sigma < 1, \\
(q|t|)^{1/2 - \sigma} \log(q|t|) & \text{if } \sigma < 0.
\end{cases}
\]

**Proof.** (1) is proved in [5, p. 276] and (2) follows from (1) easily by use of Lemma 1.

**Lemma 3.** Let \( L(s, \chi) \) be as in Lemma 1. Let

\[
G(n, \chi) = \sum_{m=1}^{q} \chi(m) \exp(2\pi i mn/q).
\]

For \( \text{Re}(s) > 1 \), let

\[
L^*(s, \chi) = \sum_{n=1}^{\infty} G(n, \chi)n^{-s}.
\]

Then \( L(s, \chi) \) and \( L^*(s, \chi) \) satisfy the functional equation

\[
L(s, \chi) = (i/q)(2\pi/q)^{s-1} \Gamma(1-s) \left\{ e^{\pi is/2} + \chi(-1)e^{-\pi is/2} \right\} L^*(1-s, \chi).
\]

This result can be found in [1, Theorems 1 and 3].

**3. Proof of the theorem.** For \( \text{Re}(s) > 1 \), if we let

\[
F(s) = \sum_{n=1}^{\infty} f(n)n^{-s},
\]

then we have

\[
F(s) = L(s, \chi_1)L(s, \chi_2).
\]
By Perron’s formula, we have, by (3.1),

\[
\sum_{n \leq x} f(n) \log(x/n) = \frac{1}{2\pi i} \int_{1+1/\log x, T} F(s)x^s s^{-2} ds + O(x^{\delta}/T + x^{1+\delta} \log x/T^2),
\]

for any fixed \( \delta > 0 \).

Let \( T > 1 \) and \( R_T \) be the rectangle with vertices: \( 1+1/\log x+iT \) and \( -1/14+iT \). Then on the interior of \( R_T F(s)x^s s^{-2} \) has a double pole at \( s = 0 \) and a double, single or no pole at \( s = 1 \), depending on whether both, one or none of the characters, \( \chi_1 \) and \( \chi_2 \), are principal characters. On \( R_T x^s F(s)s^{-2} \) is analytic. Thus, by the residue theorem, we have

\[
\frac{1}{2\pi i} \int_{R_T} F(s)x^s s^{-2} ds = \text{res}(F(s)x^s s^{-2}, 0) + \text{res}(F(s)x^s s^{-2}, 1).
\]

In a neighborhood of \( s = 0 \) we have

\[
x^s s^{-2} f(s) = s^{-2}(1 + s \log x + \frac{1}{2}s^2 + O(s^3))(F(0) + F'(0)s + O(s^2))
\]

Thus

\[
\text{res}(x^s s^{-2} F(s), 0) = F(0) \log x + F'(0).
\]

If neither character is principal, then

\[
\text{res}(x^s F(s)s^{-2}, 1) = 0.
\]

Suppose \( \chi_1 \) is a principal character modulo \( q_1 \) and \( \chi_2 \) is a nonprincipal character modulo \( q_2 \). Then, in a neighborhood of \( s = 1 \), we have

\[
L(s, \chi_1) = \gamma_{-1}(q_1)(s-1)^{-1} + \gamma(q_1) + O(s-1),
\]

\[
L(s, \chi_2) = L(1, \chi_2) + L'(1, \chi_2)(s-1) + O(s-1)^2,
\]

\[
x^s = x(1 + (s-1) \log x + O(s-1)^2)
\]

and

\[
s^2 = 1 - (s-1) + O(s-1)^2.
\]

Combining these expansions gives

\[
\text{res}(x^s F(s)s^{-2}, 1) = L(1, \chi_2)\gamma_{-1}(q_1)x.
\]

Similarly, if \( \chi_1 \) is a nonprincipal character modulo \( q_1 \) and \( \chi_2 \) is the principal character modulo \( q_2 \), then

\[
\text{res}(x^s F(s)s^{-2}, 1) = L(1, \chi_1)\gamma_{-1}(q_2)x.
\]

If both \( \chi_1 \) and \( \chi_2 \) are principal characters, then we have, as above,

\[
\text{res}(x^s F(s)s^{-2}, 1) = \gamma_{-1}(q_1)\gamma_{-1}(q_2)x \log x
\]

\[
+ (\gamma_{-1}(q_1)\gamma(q_2) + \gamma_{-1}(q_2)\gamma(q_1) - \gamma_{-1}(q_1)\gamma_{-1}(q_2))x.
\]
Let

\[ C_1(\chi_1, \chi_2) = \begin{cases} 
\gamma_1(q_1) \gamma_1(q_2) & \text{if } \chi_1 \text{ and } \chi_2 \text{ are principal,} \\
0 & \text{else,}
\end{cases} \]
\[ C_2(\chi_1, \chi_2) = \begin{cases} 
\gamma_1(q_1) \gamma_1(q_2) + \gamma_1(q_1) \gamma(q_2) + \gamma_1(q_2) \gamma(q_1) & \text{if both } \chi_1 \text{ and } \chi_2 \text{ are principal,} \\
L(1, \chi_2) \gamma_1(q_1) & \text{if } \chi_1 \text{ principal, } \chi_2 \text{ not,} \\
L(1, \chi_1) \gamma_1(q_2) & \text{if } \chi_2 \text{ principal, } \chi_1 \text{ not,} \\
0 & \text{neither character principal,}
\end{cases} \]

(3.10)

\[ C_3(\chi_1, \chi_2) = F(0) \quad \text{and} \quad C_4(\chi_1, \chi_2) = \gamma'(0). \]

Then, by (3.4)-(3.10), we have

\[ \frac{1}{2\pi i} \int_{R_T} F(s) x^s s^{-2} \, ds = C_1(\chi_1, \chi_2) x \log x + C_2(\chi_1, \chi_2) x \\
+ C_3(\chi_1, \chi_2) \log x + C_4(\chi_1, \chi_2). \]

By Lemma 2 and (3.2), we have

\[ \left| \int_{-1/14}^{1/14} x^s F(s) s^{-2} \, ds \right| = \left| \int_{1+1/\log x}^{1-1/14} x^{\sigma+iT}(\sigma \pm iT)^{-2} F(\sigma \pm iT) \, d\sigma \right| \\
= \left| \left\{ \int_{1+1/\log x}^{1} + \int_{1}^{0} \int_{0}^{-1/14} \right\} x^{\sigma+iT}(\sigma \pm iT)^{-2} F(\sigma \pm iT) \, d\sigma \right| \\
\ll x/T^2 + x \log^2 T/T + T^{6/7} \log^2 T. \]

(3.12)

To estimate the fourth side of \( R_T \), the integral

\[ \int_{(-1/14, T)} x^s F(s) s^{-2} \, ds = \int_{-T}^{T} x^{-1/14+it}(-1/14 + it)^2 F(-1/14 + it) \, dt, \]

we use the relation (3.2) and the functional equation given in Lemma 3 to rewrite the integral on the right-hand side of (3.13). This is then estimated, by Sterling's formula and a lemma of van der Corput [4, Lemma 4.5], as in [4, pp. 265–266]. The result is

\[ \int_{(-1/14, T)} x^s F(s) s^{-2} \, ds \ll T^{-6/7} + x^{-1/14}T^{-5/14}. \]

(3.14)

Thus, (3.3), (3.11), (3.12) and (3.14), we have

\[ \sum_{n \leq x} f(n) \log(x/n) = C_1(\chi_1, \chi_2) x \log x + C_2(\chi_1, \chi_2) x \\
+ C_3(\chi_1, \chi_2) \log x + C_4(\chi_1, \chi_2) \\
+ O(x^5/T + T^{-2}x^{1+6} \log x + x/T^2 + T^{-6/7} \log^2 T \\
+ x \log^2 T/T + x^{-1/14}T^{-5/14} + T^{-6/7}). \]

If we choose \( T = x^{5/3} \) we get the result of the theorem.

This completes the proof of the theorem.
4. Further computation of the coefficients $C_j(\chi_1, \chi_2)$.

LEMMA 4. Let $\chi_0$ be the principal character modulo $q$. If, in a neighborhood of $s = 1$,
$$L(s, \chi_0) = \gamma_1(q)(s - 1)^{-1} + \gamma(q) + O_q(s - 1),$$
then
$$\gamma_1(q) = \varphi(q)/q \quad \text{and} \quad \gamma(q) = \varphi(q) \left\{ \sum_{p \mid q} \frac{\log p}{p - 1} + \gamma \right\},$$
where $\varphi(q)$ is Euler's function and $\gamma$ is Euler's constant.

PROOF. We have
$$L(s, \chi_0) = \prod_{p \mid q} (1 - p^{-s}) \zeta(s).$$

In a neighborhood of $s = 1$, we have
$$\zeta(s) = (s - 1)^{-1} + \gamma + O(s - 1)$$
and
$$\prod_{p \mid q} (1 - p^{-s}) = \varphi(q) \left\{ 1 + \sum_{p \mid q} \frac{\log p}{p - 1} (s - 1) + O_q(s - 1) \right\}.$$

If we combine the representations (4.2)–(4.4), the result, (4.1), follows by comparing coefficients and completes the proof of the lemma.

LEMMA 5. Let $L(s, \chi)$ be as in Lemma 1. Then
$$L(0, \chi) = \sum_{n=1}^{q} \chi(n) \left( \frac{1}{2} - n/q \right)$$
and
$$L'(0, \chi) = \log q \sum_{n=1}^{q} \chi(n) \left( \frac{1}{2} - n/q \right) + \sum_{n=1}^{q} (\log \Gamma(n/q) - \frac{1}{2} \log 2\pi) \chi(n).$$

PROOF. By Lemma 1, we have
$$L(s, \chi) = q^{-s} \sum_{n=1}^{q} \chi(n) \zeta(s, n/q).$$

Thus
$$L'(s, \chi) = q^{-s} \log q \sum_{n=1}^{q} \chi(n) \zeta(s, n/q).$$

Thus
$$L(0, \chi) = \sum_{n=1}^{q} \chi(n) \zeta(0, n/q)$$
and
$$L'(0, \chi) = \log q \sum_{n=1}^{q} \chi(n) \zeta(0, n/q) + \sum_{n=1}^{q} \chi(n) \zeta'(0, n/q).$$

By [5, p. 271], we have
$$\zeta(0, n/q) = \frac{1}{2} - n/q \quad \text{and} \quad \zeta'(0, n/q) = \log \Gamma(n/q) - \frac{1}{2} \log 2\pi.$$
Combining these results gives the lemma and completes the proof. Thus, by (3.2), (3.10) and Lemmas 4 and 5, we have

$$C_1(x_1, x_2) = \begin{cases} \phi(q_1) \phi(q_2) / q_1 q_2 & \text{if both characters are principal,} \\ 0 & \text{else,} \end{cases}$$

$$C_2(x_1, x_2) = \begin{cases} \phi(q_1) \phi(q_2) \left( \left( \sum_{p | q_1} + \sum_{p | q_2} \right) \frac{\log p}{p - 1} + 2 \gamma - 1 \right) & \text{if both characters are principal,} \\ L(1, \chi_2) \phi(q_1) / q_1 & \text{if } \chi_1 \text{ is principal, } x_2 \text{ not,} \\ L(1, \chi_1) \phi(q_2) / q_2 & \text{if } x_2 \text{ is principal, } x_1 \text{ not,} \\ 0 & \text{else,} \end{cases}$$

\[(4.5)\]

$$C_3(x_1, x_2) = \sum_{n=1}^{q_1} x_1(n) \left( \frac{1}{q_1} - n / q_1 \right) \sum_{n=1}^{q_2} x_2(n) \left( \frac{1}{q_2} - n / q_2 \right), \quad \text{and}$$

$$C_4(x_1, x_2) = \sum_{n=1}^{q_1} x_1(n) \left( \frac{1}{q_1} - n / q_1 \right) \sum_{n=1}^{q_2} x_2(n) \left( \log \Gamma(n / q_2) - \frac{1}{2} \log 2 \pi \right)$$

$$+ \sum_{n=1}^{q_1} x_1(n) \left( \log \Gamma(n / q_1) - \frac{1}{2} \log 2 \pi \right) \sum_{n=1}^{q_2} x_2(n) \left( \frac{1}{q_2} - n / q_2 \right)$$

$$+ \left( \log q_1 q_2 \right) C_3(x_1, x_2).$$

5. Examples.

**Example 1.** Let $x_1 = x_2 = 1$ and $q_1 = q_2 = 1$. Then $f(n) = d(n)$, the divisor function. The theorem gives

$$\sum_{n \leq x} f(n) \log(x / n) = x \log x + (2 \gamma - 1) x + \frac{1}{4} \log x + \frac{1}{2} \log 2 \pi + O(x^{-2/3}),$$

as $x \to \infty$.

**Example 2.** Let $x_1 \neq 1$, $q_1 = 1$ and $q_2 = 4$ with $x_2$ the nonprincipal character modulo 4. Then $f(n) = \frac{1}{4} r(n)$, where $r(n)$ is the number of representations of $n$ as a sum of two squares. The theorem gives, as $x \to \infty$,

$$\sum_{n \leq x} r(n) \log(x / n) = \pi x + \log x - \log(\pi / 2) - 2 \log(\Gamma(\frac{1}{4}) / \Gamma(\frac{3}{4})) + O(x^{-2/3}).$$

This betters the result of Müller [3] who obtained an error term of $O(x^{-1/4})$.

**Example 3.** Let $q_1 = 1$, $x_1 \equiv 1$ and $x_2$ be the Kronecker character modulo $-q$. Then $f(n)$ is the number of integral ideals of norm $n$ in the imaginary quadratic field $\mathbb{Q}(\sqrt{-q})$. The theorem gives, as $x \to \infty$,

$$\sum_{n \leq x} f(n) \log(x / n) = L(1, \chi_2) x + q^{-1} \sum_{n=1}^{q} n(q / n) \log x$$

$$- \left( \frac{1}{2} \log 2 \pi \right) q^{-1} \sum_{n=1}^{q} n(q / n) + (\log q / 2q) \sum_{n=1}^{q} n(q / n)$$

$$- \frac{1}{2} \sum_{n=1}^{q} (q / n) \log \Gamma(n / q) + O(x^{-2/3}),$$

where $(q / n)$ is the Kronecker character. This betters the result of Ayoub and Chowla [2] who obtained an error term of $O(x^{-1/4})$.  

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