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QUANTITATIVE BEHAVIOUR OF THE NORMS OF AN ANALYTIC MEASURE

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ABSTRACT. A Littlewood-Paley type inequality for the quotient norms of an analytic measure is obtained; one consequence of this inequality is the classical theorem of F. and M. Riesz.

In this paper $\mathbf{T}$ is the circle group, $\mathbf{Z}$ the additive group of integers and $M(\mathbf{T})$ the customary space of Borel measures on $\mathbf{T}$; for $\mu \in M(\mathbf{T})$ and $n \in \mathbf{Z}$ define

$$\hat{\mu}(n) = \int_{\mathbf{T}} e^{-in\theta} \, d\mu(\theta).$$

If $\mu \in M(\mathbf{T})$, put $\mu = \mu_a + \mu_s$ where μ_a is absolutely continuous with respect to Lebesgue measure on $\mathbf{T}$ and μ_s is singular with respect to Lebesgue measure on $\mathbf{T}$; let $M_a(\mathbf{T})$ denote the space of all absolutely continuous measures. A measure $\mu \in M(\mathbf{T})$ is said to be of *analytic type* if $\hat{\mu}(n) = 0$ for all $n < 0$; as usual $H^1(\mathbf{T})$ denotes the space of all analytic measures on $\mathbf{T}$.

F. and M. Riesz [7] have proved the following result for measures of analytic type.

THEOREM R. *If* $\mu \in H^1(\mathbf{T})$ *then* $\mu \in M_a(\mathbf{T})$.

The purpose of this paper is to present a quantitative version of the F. and M. Riesz Theorem; before stating our result, we will need two quantitatively explicit lemmas concerning the behaviour of the quotient norms of a measure.

For any $\omega \in M(\mathbf{T})$ and $E \subset \mathbf{Z}$ put

$$\|\omega\|_E = \inf\{\|\nu\| \colon \hat{\nu} = \hat{\omega} \text{ on } E, \nu \in M(\mathbf{T})\};$$

here $\|\cdot\|$ is the usual total variation norm for $M(\mathbf{T})$. The following lemma provides an effective means of computing quotient norms.

LEMMA A. *For any* $\omega \in M(\mathbf{T})$ *and* $E \subset \mathbf{Z}$

$$\|\omega\|_E = \sup\left\{\left|\sum_{-m}^{m} c_n \hat{\omega}(n)\right| : \left\|\sum_{-m}^{m} c_n e^{-in\theta}\right\|_\infty \leq 1, n \in E \text{ for all } n\right\}.$$

PROOF. Fix $\omega \in M(\mathbf{T})$ and $E \subset \mathbf{Z}$; that

$$\sup\left\{\left|\sum_{-m}^{m} c_n \hat{\omega}(n)\right| : \left\|\sum_{-m}^{m} c_n e^{-in\theta}\right\|_\infty \leq 1, \ n \in E \text{ for all } n\right\} \leq \|\omega\|_E$$

is obvious; to confirm the reverse inequality simply apply the Hahn-Banach and Riesz representation theorems.

The next lemma describes the behaviour at infinity of the quotient norms of a measure.

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LEMMA L. *For any natural number n put* $A_n = [n,\infty) \cup (-\infty, -n] \subset \mathbf{Z}$; *if* $\omega \in M(\mathbf{T})$ *then* $\lim_{n\to\infty} \|\omega\|_{A_n} = \|\omega_s\|$.

PROOF. Lemma L is an immediate consequence of Lemma A and the work of R. Doss [**3**].

We say $\langle D_n\rangle_1^\infty$ is a sequence of *positive dyadic intervals* in $\mathbf{Z}$ if there exists a sequence $\langle n_k\rangle_1^\infty \subset \mathbf{Z}^+$, $n_{k+1}/n_k \geq 2$ for all k and $D_k = [n_{2k}, n_{2k+1}) = \{m \in \mathbf{Z}: n_{2k} \leq m < n_{2k+1}\}$. We say $\langle D_n\rangle_1^\infty$ is a sequence of *symmetric dyadic intervals* in $\mathbf{Z}$ if there exists a sequence $\langle n_k\rangle_1^\infty \subset \mathbf{Z}^+$, $n_{k+1}/n_k \geq 2$ for all k and $D_k = [n_{2k}, n_{2k+1}) \cup (-n_{2k+1}, -n_{2k}] \subset \mathbf{Z}$.

Our quantitative generalization of the F. and M. Riesz Theorem is then

THEOREM V. *There is a constant* $C > 0$ *such that for any sequence* $\langle D_n\rangle_1^\infty$ *of symmetric dyadic intervals in* $\mathbf{Z}$ *and any* $\mu \in H^1(\mathbf{T})$

$$\sum_{k=1}^{\infty} \frac{\|\mu\|_{D_k}}{k} \leq C\|\mu\|.$$

The statement of Theorem V occurred to the authors after a reading of [**4**]; notice that Theorem R is an immediate consequence of Theorem V and Lemma L. For results related to Theorem V see comment (e) at the end of this paper. The proof of Theorem V depends upon this result:

LEMMA N (cf. [**1, 2, 6**]). *Let* $\langle a_n\rangle_1^Q$ *be a sequence of complex numbers and* $\langle t_n\rangle_1^Q$ *a sequence of trigonometric polynomials on* $\mathbf{T}$ *such that* $\sum_1^Q |a_n| \leq 1$, $\|t_n\|_\infty \leq 1$ *for* $n = 1, 2, \ldots, Q$, *and* $\operatorname{supp}\hat{t}_i \cap \operatorname{supp}\hat{t}_j = \varnothing$ *for* $i \neq j$. *Put* $\beta = \sum_1^Q |a_n|^2$. *Then given* $\epsilon > 0$, *there exists a polynomial* $p(z) = b_1 z + \cdots + b_m z^m$ *such that*

$$\frac{1}{5}\left|\sum_1^Q a_n t_n\right| + \left|\exp\left(-\frac{\sqrt{2}}{4}\beta^{1/2}\right) + p\left(\left\{\sum_{i\neq j}(a_i t_i)(\overline{a_j t_j})\right\}^+\right)\right| < 1 + \epsilon;$$

here $(\sum_{i\neq j}(a_i t_i)(\overline{a_j t_j}))^+$ *is the analytic projection of* $\sum_{i\neq j}(a_i t_i)(\overline{a_j t_j})$.

PROOF OF LEMMA N. Put $f = \sum_1^Q a_i t_i$; then

$$|f|^2 = \sum |a_i|^2 |t_i|^2 + 2\operatorname{Re}\left\{\left(\sum_{i\neq j} a_i t_i \overline{a_j t_j}\right)^+\right\}.$$

Given $\epsilon > 0$ choose $\delta > 0$ so that

$$\left|\exp\left(-\frac{\sqrt{2}}{4}(\beta+\delta)^{1/2}\right) - \exp\left(-\frac{\sqrt{2}}{4}\beta^{1/2}\right)\right| < \frac{\epsilon}{2} \tag{1.1}$$

and define $h = \beta + \delta + 2(\sum_{i\neq j} a_i t_i \overline{a_j t_j})^+$; notice that $\operatorname{Re} h \geq |f|^2 + \delta$ because $\|t_i\|_\infty \leq 1$ $(i = 1, 2, \ldots, Q)$. We gather from this that there exists a disc $D_R \subset \mathbf{C}$ centered about $R > 0$ which lies entirely in the open right half plane such that the range of h is contained in D_R and $\beta + \delta \in D_R$.

Put $H(z) = z^{1/2}$; then $\exp(-\sqrt{2}H(z)/4)$ is analytic on D_R and so it follows that there is a polynomial $F(z) = c_0 + c_1(z - R) + \cdots + c_m(z - R)^m$ such that for all

$z \in D_R$

$$(1.2)\qquad \left|\exp\left(-\frac{\sqrt{2}}{4}H(z)\right) - F(z)\right| < \frac{\epsilon}{4}.$$

Since $\beta + \delta \in D_R$ and Range $h \subset D_R$ we infer via inequalities (1.1) and (1.2) that

$$(1.3)\qquad \left|\exp\left(-\frac{\sqrt{2}}{4}H(h)\right) - \exp\left(-\frac{\sqrt{2}}{4}\beta^{1/2}\right) - p\left(\sum_{i\neq j}(a_it_i)(\overline{a_jt_j})^+\right)\right| < \epsilon$$

for some polynomial of the form $p(z) = b_1z + b_2z^2 + \cdots + b_mz^m$.

To complete the proof of our lemma observe that

$$\left|\exp\left(-\frac{\sqrt{2}}{4}H(h)\right)\right| + \frac{1}{5}\left|\sum_1^Q a_it_i\right| \leq \exp\left(-\frac{\sqrt{2}}{4}\operatorname{Re}H(h)\right) + \frac{1}{5}|f|$$
$$\leq \exp\left(-\frac{|f|}{4}\right) + \frac{1}{5}|f|$$

because $\operatorname{Re}H(h) \geq |f|/\sqrt{2}$. Inasmuch as $\exp(-x/4) + x/5 \leq 1$ for $0 \leq x \leq 1$, we have confirmed

$$(1.4)\qquad \left|\exp\left(-\frac{\sqrt{2}}{4}H(h)\right)\right| + \frac{1}{5}\left|\sum_1^Q a_it_i\right| \leq 1.$$

The conclusion of our lemma now follows from inequalities (1.3) and (1.4).

We now turn to the proof of Theorem V: Let $\langle D_k\rangle_1^\infty$ be any sequence of symmetric dyadic intervals in $\mathbf{Z}$ and let $\mu \in H^1(\mathbf{T})$. Put $a_k = 1/k$ $(k = 1, 2, \ldots)$; by breaking the sequence $\langle a_k\rangle_1^\infty$ into the blocks $\{a_1\}$, $\{a_2, a_3\}$, $\{a_4, a_5, a_6, a_7\}, \ldots$, we may choose $\{Q_1 < Q_2 < \cdots\} \subset \mathbf{Z}^+$ as follows:

$$(1)\qquad \{a_1^2 + a_2^2 + \cdots + a_{Q_1}^2\}^{1/2} + \{a_{Q_1+1}^2 + \cdots + a_{Q_2}^2\}^{1/2} + \cdots \leq \frac{1}{1 - (\sqrt{2})^{-1}}$$

and

$$(2)\qquad a_1 + a_2 + \cdots + a_{Q_1} \leq 1, \quad a_{Q_1+1} + \cdots + a_{Q_2} \leq 1, \quad \text{etc.}$$

Let t_k satisfy $\operatorname{supp}\hat{t}_k \subset D_k$, $\|t_k\|_\infty \leq 1$ with $\int_{\mathbf{T}} t_k(\theta)\,d\mu(\theta) \equiv \hat{\mu}(t_k) = \|\mu\|_{D_k}$. Given $\epsilon > 0$, $\langle a_k\rangle_{Q_j+1}^{Q_{j+1}}$ and $\langle t_k\rangle_{Q_j+1}^{Q_{j+1}}$, define $\beta_j = \sum_{Q_j+1}^{Q_{j+1}} a_k^2$ $(j = 1, 2, \ldots)$ and put $\alpha_j = \exp(-\sqrt{2}\beta_j^{1/2}/4)$; then (using (2)) select polynomials of the form $p_j = b_{j1}z + \cdots + b_{jm}z^m$ such that

$$(3)\qquad \frac{1}{5}\left|\sum_{Q_j+1}^{Q_{j+1}} a_kt_k\right| + \left|\alpha_j + p_j\left(\left\{\sum_{Q_j+1\leq i\neq j\leq Q_{j+1}}(a_it_i)(\overline{a_jt_j})\right\}^+\right)\right| \leq 1 + \epsilon^{2^{j-1}}.$$

Such a selection of polynomials p_j is always possible by Lemma N.

Put $F_0 = \frac{1}{5}\sum_{k=1}^{Q_1} a_kt_k$ and for $n = 1, 2, \ldots$ define

$$F_n = \frac{1}{5}\sum_{Q_n+1}^{Q_{n+1}} a_kt_k + F_{n-1}\left\{\alpha_n + p_n\left(\left\{\sum_{Q_n+1\leq i\neq j\leq Q_{n+1}}(a_it_i)(\overline{a_jt_j})\right\}^+\right)\right\};$$

then it is not hard to see that condition (3) implies

$$\|F_n\|_\infty \le \frac{1}{1-\epsilon} \qquad (n = 1, 2, \ldots).$$

So, on the one hand,

$$\left|\int_{\mathbf{T}} F_n(\theta)\, d\mu(\theta)\right| \le \frac{1}{1-\epsilon}\|\mu\|, \tag{4}$$

while on the other,

$$\begin{aligned} 5\int_{\mathbf{T}} F_n(\theta)\, d\mu(\theta) = {} & \sum_{Q_n+1}^{Q_{n+1}} a_k\hat{\mu}(t_k) + \alpha_n \sum_{Q_{n-1}+1}^{Q_n} a_k\hat{\mu}(t_k) \\ & + \alpha_n\alpha_{n-1} \sum_{Q_{n-2}+1}^{Q_{n-1}} a_k\hat{\mu}(t_k) + \cdots + (\alpha_n\alpha_{n-1}\cdots\alpha_1)\sum_{1}^{Q_1} a_k\hat{\mu}(t_k), \end{aligned} \tag{5}$$

because $\mu \in H^1(\mathbf{T})$ and the sequence $\langle D_k\rangle_1^\infty$ is dyadic. As a consequence of (4) and (5), we obtain

$$\frac{5}{1-\epsilon}\|\mu\| \ge (\alpha_n\alpha_{n-1}\cdots\alpha_1)\left\{\sum_{Q_n+1}^{Q_{n+1}} a_k\hat{\mu}(t_k) + \cdots + \sum_{1}^{Q_1} a_k\hat{\mu}(t_k)\right\}. \tag{6}$$

It now follows from inequalities (1) and (6) that

$$5\exp(2\sqrt{2}-2)^{-1}\cdot\|\mu\| \ge (1-\epsilon)\sum_{1}^{Q_{n+1}} a_k\hat{\mu}(t_k); \tag{7}$$

consequently $17\|\mu\|/(1-\epsilon) \ge \sum_1^{Q_{n+1}} \frac{1}{k}\hat{\mu}(t_k)$, and this, in turn, implies that

$$\sum_{1}^{\infty} \frac{1}{k}\|\mu\|_{D_k} \le 17\|\mu\|.$$

COMMENTS. (a) As the proof of Theorem V shows, we can relax the gap condition on the intervals somewhat, and also change the sequence $\langle 1/k\rangle_1^\infty$ to any sequence $\langle a_k\rangle_1^\infty \subset \mathbf{Z}^+$ satisfying conditions analogous to (1) and (2).

(b) The obvious analogue of Theorem V remains valid for the additive group of real numbers $\mathbf{R}$.

(c) Let G be a compact abelian group with character group Γ; suppose $\phi\colon \Gamma \to \mathbf{R}$ is a nontrivial homomorphism. We say $\langle {}_\phi D_k\rangle_1^\infty$ is a sequence of *symmetric ϕ-dyadic intervals* in Γ if there exists a sequence $\langle D_k\rangle_1^\infty$ of symmetric dyadic intervals in $\mathbf{Z}$ such that for each k ${}_\phi D_k = \phi^{-1}(D_k)$. *Then, there is a constant $C > 0$ such that for any sequence $\langle {}_\phi D_k\rangle_1^\infty$ of symmetric ϕ-dyadic intervals in Γ and any $\mu \in H^1_\phi(G) = \{\nu \in M(G)\colon \hat{\nu}(\gamma) = 0$ for all $\gamma \in \Gamma$ such that $\phi(\gamma) < 0\}$*

$$\sum_{k=1}^{\infty} \frac{\|\mu\|_{{}_\phi D_k}}{k} \le C\|\mu\|.$$

An immediate consequence of the above inequality is that if $\mu \in H^1_\phi(G)$ then μ translates continuously in the direction of ϕ and this in turn implies that $\mu_s \in H^1_\phi(G)$; see [6] for definitions and appropriate references.

(d) We say $\langle D_k \rangle_1^\infty$ is a sequence of *symmetric dyadic boxes* in $\mathbf{Z}^n$ if there exists a sequence $\langle m_k \rangle_1^\infty \subset \mathbf{Z}^+$, $m_{k+1}/m_k \geq 2$ for all k and

$$D_k = \{(y_1, \ldots, y_n) \colon |y_i| \in [m_{2k}, m_{2k+1}) \cup \{0\}\} \backslash \{0\} \subset \mathbf{Z}^n.$$

Then, there is a constant $C > 0$ such that for any sequence $\langle D_k \rangle_1^\infty$ of symmetric dyadic boxes in $\mathbf{Z}^n$ and any $\mu \in M(\mathbf{T}^n)$

$$\sum_1^\infty \frac{\|\mu\|_{D_k}}{k} \leq C\|\mu\|$$

provided $\operatorname{supp} \hat{\mu} \subset \{(x_1, \ldots, x_n) \colon x_i \in \mathbf{Z},\ x_i \geq 0 \text{ for all } i\}$. It is not hard to see that the above inequality is a quantitative generalization of S. Bochner's several-variable extension of the F. and M. Riesz Theorem.

To obtain the proofs of (c) and (d) it suffices to repeat the proof of Theorem V after making appropriate modifications in Lemmas A, L, and N.

(e) A sequence of symmetric dyadic intervals $\langle D_{k_1} \rangle^\infty$, $D_k = [n_{2k}, n_{2k+1}) \cup (-n_{2k+1}, -n_{2k}]$, is said to be *standard* if there exists a $\lambda > 0$ such that $n_{k+1}/n_k < \lambda$ for all k. Notice that Theorem V for standard dyadic intervals is a special case of known Littlewood-Paley type inequalities [8]; however, in order to obtain our applications it is important that we not restrict ourselves to standard dyadic intervals.

The original proof of Theorem V used a linear fractional transformation rather than the exponential function of the present argument; the use of the exponential function makes more explicit the connection between the method of proof of Theorem V and [5].

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