PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 4, December 1982

REMOVABLE SINGULARITIES FOR H^p-FUNCTIONS

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ABSTRACT. Given a domain D in $\mathbf{C}^n$, a holomorphic function f on D is said to belong to $H^p(D)$, $0 < p < \infty$, provided that $|f|^p$ admits a harmonic majorant in D. In this note it is shown that $H^p(D \backslash E) = H^p(D)$ whenever E is a relatively closed polar subset of D.

1. According to a theorem of M. Parreau [**4**, Théorème 20], closed sets of logarithmic capacity zero are removable singularities for H^p-functions on Riemann surfaces, i.e., given any Riemann surface W, $H^p(W \backslash E) = H^p(W)$ for each $p > 0$ whenever $E \subset W$ is a closed set of capacity zero. Recently, J. Cima and I. Graham [**1**] showed that for certain subdomains of $\mathbf{C}^n$, analytic subvarieties are inessential singularities in the same sense. The purpose of the present note is to point out that Parreau's result extends in full generality for subdomains of $\mathbf{C}^n$; in other words, for each domain $D \subset \mathbf{C}^n, H^p(D \backslash E) = H^p(D)$ whenever E is a relatively closed polar subset of D. Our method of proof has its origin in the cited work of Parreau.

2. Let $D \subset \mathbf{C}^n$ be a domain, and let f be a holomorphic function on D. For $0 < p < \infty$, f is said to belong to $H^p(D)$ if $|f|^p$ has a harmonic majorant in D (cf. [**6**, pp. 37–38]), and $f \in H^\infty(D)$ whenever f is bounded.

A set $E \subset \mathbf{C}^n$ $(= R^{2n})$ is a *polar* set if there is an open set $D \subset \mathbf{C}^n$ containing E and a function u subharmonic on D such that $u = -\infty$ on E. If, in particular, E is a closed polar subset of a domain $D \subset \mathbf{C}^n$, then E is nowhere dense and $D \backslash E$ is connected; further, an analytic subvariety of D is always a polar set [**3**, p. 34]. In case $n = 1$, E is a polar set if and only if E is of logarithmic capacity zero.

A nonnegative harmonic function u on a domain $D \subset \mathbf{C}^n$ is said to be *quasi-bounded* [**4**, p. 165; **2**, p. 7] provided that there exists a nondecreasing sequence of nonnegative bounded harmonic functions on D which has limit u; and u is *singular* provided that the only nonnegative bounded harmonic function on D majorized by u is the constant zero. By [**4**, Théorème 12], each nonnegative harmonic function h on D admits a unique representation of the form $u = q + s$, where q is quasi-bounded and s is singular; see also [**2**, p. 7]. It is to be noted that the mapping $u \mapsto q$ is additive and homogeneous [**4**, p. 165].

REMARK. Although the work of Parreau and Heins is performed in the context of Riemann surfaces, their arguments apply verbatim to the situation considered here.

We shall make use of the following lemma; a proof can be found in [**2**, pp. 18–19].

LEMMA. *Let $D \subset \mathbf{C}^n$ be a domain, and let $u\colon D \to [-\infty, \infty)$ be subharmonic on D. Let $\varphi\colon [-\infty, \infty) \to [0, \infty)$ be a continuous, nondecreasing function such that $\varphi|\mathbf{R}$*

Received by the editors December 22, 1981 and, in revised form, March 22, 1982.
1980 *Mathematics Subject Classification*. Primary 32D20; Secondary 32A35.
Key words and phrases. H^p-function, polar set, quasi-bounded harmonic function.

is convex and $\lim_{x\to\infty} x^{-1}\varphi(x) = \infty$. *Suppose that* $\varphi \circ u$ *has a harmonic majorant. Then* $M(\varphi \circ u)$, *the least harmonic majorant of* $\varphi \circ u$, *is quasi-bounded.*

Note that $\varphi \circ u$ also is subharmonic on D.

3. We are now in a position to establish a generalization of [**1**, Theorem A].

THEOREM 1. *Let* $D \subset \mathbf{C}^n$ *be a domain, let* E *be a relatively closed polar subset of* D, *let* $p \in (0,\infty]$, *and let* $f \in H^p(D\backslash E)$. *Then* f *admits a unique holomorphic extension* f^* *to* D, *and* $f^* \in H^p(D)$.

PROOF. For $p = \infty$, the result is due to Lelong [**3**, p. 35]. So assume that $p \in (0,\infty)$, and let u stand for the least harmonic majorant of $|f|^p$ on $D\backslash E$. Since $\log|f|$ is subharmonic on $D\backslash E$—we assume that $f \not\equiv 0$—and the function

$$x \mapsto e^{px} \quad \text{for } x \in (-\infty,\infty), \qquad -\infty \mapsto 0,$$

satisfies the assumptions of the Lemma, we infer that u is quasi-bounded. By definition, we can find a sequence (u_n) of bounded harmonic functions on $D\backslash E$ which converges nondecreasingly towards u. Thanks to a well-known extension theorem in potential theory (see e.g. [**3**, p. 35]), each u_n can be uniquely continued to become harmonic and bounded on D. It is clear that (u_n^*), the sequence of the extended functions, is nondecreasing; hence, by Harnack's theorem, the limit function u^* gives a harmonic extension of u to D. It follows that f is locally bounded on D; i.e., for each $p \in D$ there is a neighborhood $U_p \subset D$ of p such that $f|U_p\backslash E$ is bounded. Invoking the result of Lelong mentioned above, we see that $f|U_p\backslash E$ extends to a holomorphic function on U_p for each p. Altogether, we have a function f^*, holomorphic on D, such that $f^*|D\backslash E = f$ and $|f^*|^p$ is majorized by u^*.

REMARK. A proof of Parreau's theorem which proceeds along these lines can be found in [**7**].

4. For certain domains in $\mathbf{C}^n$, it seems, while defining Hardy spaces, to be more natural to consider n-harmonic majorants instead of harmonic majorants; for instance, this is the case with the unit polydisc $U^n \subset \mathbf{C}^n$ [**5**, Chapter 3; **1**]. However, we insist on the definition given at the outset of this paper and therefore formulate our version of [**1**, Theorem B] as follows. As for the terminology: a continuous real (or complex) function u in a domain $D \subset \mathbf{C}^n$ is *n-harmonic* if u is harmonic in each (complex) variable separately; *n-subharmonic* and *n-superharmonic* functions are defined analogously [**5**, p. 39]; a set $X \subset \mathbf{C}^n$ is said to be a polar set for n-subharmonic functions, in short *n-polar*, provided that there is an open set D containing X and a function u n-subharmonic on D such that $u = -\infty$ on X [**1**, p. 246]. Note that analytic subvarieties of a domain in $\mathbf{C}^n$ are n-polar sets.

THEOREM 2. *Let* E *be a relatively closed* n*-polar subset of* U^n. *Let* f *be a holomorphic function on* $U^n\backslash E$ *such that for some* $p \in (0,\infty)$, $|f|^p$ *has an* n*-harmonic majorant* u *on* $U^n\backslash E$. *Then* f *extends to a holomorphic function* f^* *on* U^n *such that* $|f^*|^p$ *has an* n*-harmonic majorant on* U^n.

PROOF. By Theorem 1, there is a unique holomorphic function f^* on U^n such that $f^* \mid U^n\backslash E = f$. Let u^* denote the unique n-superharmonic extension of u to U^n; this exists by [**1**, Theorem 3.11]. Since $|f^*|^p \le u^*$ on U^n (cf. the proof of [**1**, Theorem 3.11]), we have for each $r \in (0,1)$

$$\int_{T^n} |f^*(rz)|^p \, dm_n \le \int_{T^n} u^*(rz) \, dm_n, \tag{1}$$

where T^n denotes the distinguished boundary of U^n [**5**, p. 3] and dm_n the normalized Lebesgue measure on T^n. We now argue as in [**1**, Theorem B]. By n-subharmonicity of $|f^*|^p$, the left-hand side of (1) is an increasing function of r [**5**, p. 40]. Since the right-hand side in turn is a decreasing function of r, it follows that

$$\int_{T^n} |f^*(rz)|^p \, dm_n, \qquad 0 < r < 1.$$

is uniformly bounded. But this property is known to be an equivalent to existence of an n-harmonic majorant for $|f^*|^p$ on U^n [**1**, p. 243; **5**, pp. 52–53].

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