PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 4, December 1982

FINITE TYPE FUNCTIONS AS LIMITS OF EXPONENTIAL SUMS

D. J. NEWMAN[1]

ABSTRACT. The functions $f(z)$ of type 1 which are bounded on the real axis are usually thought of as spanned by the exponentials $e^{i\lambda z}$, $\lambda \in [-1, 1]$. We find the correct topology in which this "spanning" occurs.

The class of functions of finite exponential type, say type 1, which are bounded on the real axis is generally thought of as being "spanned" by the exponential functions $e^{i\lambda z}$, $\lambda \in [-1, 1]$. Apparently, however, the "correct" topology for this spanning has not been recorded. Thus it is known that being of the form $\int_{-1}^{1} e^{i\lambda z} d\mu(\lambda)$ is sufficient for membership in the class, but not necessary, while being of the form $\lim_{n\to\infty} \int_{-1}^{1} e^{i\lambda z} d\mu_n(\lambda)$ is necessary, but not sufficient. We show that "weak" convergence is the proper topology.

DEFINITIONS. We say that a sequence of functions, $f_n(x)$, *converges weakly* to $f(x)$ or, in symbols, $f_n(x) \overset{w}{\to} f(x)$ when

(1) $f_n(x) \to f(x)$ uniformly on all finite intervals, and

(2) $|f_n(x)| \leqslant M$, M fixed, for all n and all real x.

The class A_1 consists of all functions which are weak limits of finite exponential sums, $\Sigma C_\lambda e^{i\lambda x}$ wherein all λ lie in $[-1, 1]$.

The class B_1 consists of all functions, $f(z)$, which are entire and satisfy the growth condition $|f(x + iy)| \leqslant Me^{|y|}$, M fixed, for all real x and y. (Equivalently B_1 is the class of type 1 functions which are bounded on the real axis.)

THEOREM. $A_1 = B_1$.

PROOF. I: $A_1 \subseteq B_1$.

We need the following

LEMMA. *Let $f(z)$ be analytic in* $\operatorname{Im} z \geqslant 0$ *and have its modulus bounded by* 1 *there. If* $|f(z)| \leqslant \varepsilon$ *on the interval* $[-A, A]$ *then* $|f(z)| \leqslant \sqrt{\varepsilon}$ *throughout the half disc* $|z| \leqslant A$, $\operatorname{Im} z \geqslant 0$.

PROOF. Fix $u + iv$ inside $|z| \leqslant A$, $\operatorname{Im} z \geqslant 0$, and consider the linear fractional transformation $z \to u - v^2/(z - u)$. This is a map of $|z| \geqslant |u + iv|$ onto $|z| \leqslant |u + iv|$ so that we always either have $|z| \leqslant |u + iv| \leqslant A$ or $|u - v^2/(z - u)| \leqslant |u + iv| \leqslant A$. Hence if we form the auxiliary function $g(z) = f(z)f(u - v^2/(z - u))$ we

Received by the editors February 8, 1982.

1980 *Mathematics Subject Classification*. Primary 30D15, 30D10.

[1]Sponsored by the United States Army under Contract No. DA-31-124-ARO-D-462.

conclude that it is bounded by $1 \cdot \varepsilon$ on the real axis. Since it is clear that $g(z)$ is bounded, analytic in $\operatorname{Im} z \geqslant 0$, $z \neq u$ the maximum modulus theorem applies. Thus $|g(z)| \leqslant \varepsilon$ throughout $\operatorname{Im} z \geqslant 0$, and setting $z = u + iv$ gives $|(f(u+iv))^2| \leqslant \varepsilon$ which proves the lemma.

Now let $f_n(x) \overset{w}{\to} f(x)$ where each $f_n \in B_1$. Fix an $\varepsilon > 0$ and an interval $[-A, A]$ and choose m, n so large that $|f_m(x) - f_n(x)| \leqslant \varepsilon$ throughout $[-A, A]$. Since we also have $|f_m(x) - f_n(x)| \leqslant 2M$ throughout $(-\infty, \infty)$ and since $(f_m(z) - f_n(z))e^{iz}$ is bounded analytic in the upper half plane the lemma applies and gives

$$|(f_m(z) - f_n(z))e^{iz}| \leqslant \sqrt{2M\varepsilon}$$

throughout $|z| \leqslant A$, $\operatorname{Im} z \geqslant 0$. Thus we obtain $|f_m(z) - f_n(z)| \leqslant \sqrt{2M\varepsilon}\, e^A$ throughout this upper half disc and in like manner it can be seen to hold in the lower half disc.

Therefore $f_n(z)$ converges uniformly on compact sets and so $\lim_n f_n(z)$ is entire. In other words $f(x)$ is (continuable to be) entire. Also since $|f_n(x+iy)| \leqslant Me^{|y|}$ we find that $|f(x+iy)| \leqslant Me^{|y|}$. In short $f \in B_1$.

All in all we have proven that B_1 is closed under taking weak limits and so, since each sum $\Sigma c_\lambda e^{i\lambda x}$, $-1 < \lambda < 1$, is clearly in B_1, we conclude that $A_1 \subseteq B_1$ as required.

II: $B_1 \subseteq A_1$.

LEMMA. *If $f(x) \in B_1$ and $K_\varepsilon(t) = (\cos t - \cos(1+\varepsilon)t)/\pi\varepsilon t^2$, $\varepsilon > 0$, then we have, identically,*

$$f(x) = \int_{-\infty}^{\infty} f(x-t) K_\varepsilon(t)\, dt.$$

PROOF. Since $\int_{-\infty}^{\infty} K_\varepsilon(t)\, dt = 1$ we may write this as

$$0 = \int_{-\infty}^{\infty} [f(x) - f(x-t)] K_\varepsilon(t)\, dt$$

which is to say

$$0 = \int_{-\infty}^{\infty} \frac{f(x) - f(x-t)}{t} \frac{e^{it} - e^{i(1+\varepsilon)t}}{t} dt + \int_{-\infty}^{\infty} \frac{f(x) - f(x-t)}{t} \frac{e^{-it} - e^{-i(1+\varepsilon)t}}{t} dt.$$

Both of these integrals converge and we move the path of integration to $\operatorname{Im} t = A$ for the first integral and to $\operatorname{Im} t = -A$ for the second ($A > 0$). The first integral is estimated by

$$\int_{Ai-\infty}^{Ai+\infty} \frac{(M + Me^A)(e^{-A} + e^{-(1+\varepsilon)A})}{|t|^2} dt \leqslant \int_{Ai-\infty}^{Ai+\infty} \frac{2Me^A \cdot 2e^{-A}}{|t|^2} dt = 4M \int_{Ai-\infty}^{Ai+\infty} \frac{dt}{|t|^2} = \frac{4M\pi}{A},$$

and similarly for the second. Our identity therefore follows upon letting $A \to \infty$.

To prove II we use the fact that our weak convergence, for B_1 functions, is the same as the weak* convergence of the functions interpreted as linear functionals on

$L^1(-\infty, \infty)$. Actually weak* convergence only implies pointwise, a.e., convergence rather than the locally uniform convergence we require, but for B_1 functions we have Bernstein's theorem that $\operatorname{Sup}|f'(x)| \leqslant \operatorname{Sup}|f(x)|$ so that pointwise implies locally uniform convergence. This same reasoning allows us to equate the topology with sequential convergence.

Having identified weak convergence, then, as weak* convergence over L^1 we may deduce the "density" of the exponential polynomials by the standard functional analysis methods.

In short let $g(x)$ be any function in $L^1(-\infty, \infty)$ which satisfies $\int_{-\infty}^{\infty} g(x)e^{i\lambda x}\,dx = 0$ for all $\lambda \in [-1, 1]$. We need only show that, for any $f \in B_1$, $\int_{-\infty}^{\infty} g(x)f(x)\,dx = 0$. To do so we consider $\int_{-\infty}^{\infty} g(x(1+\varepsilon))f(x)_{\infty}\,dx$ and apply our identity. This gives

$$\int_{-\infty}^{\infty} g(x(1+\varepsilon))f(x)\,dx = \int\int g(x(1+\varepsilon))f(t)K_\varepsilon(x-t)\,ds\,dx$$

and since $g \in L^1(dx)$, $K \in L^1(dt)$, and f is bounded, we may reverse the order of integration to conclude that this in turn equals

$$\int f(t)\int g(x(1+\varepsilon))K_\varepsilon(x-t)\,dx\,dt.$$

Next, by direct integration, we have

$$K_\varepsilon(u) = \frac{1}{2\pi}\int_{-(1+\varepsilon)}^{(1+\varepsilon)} \tau(S)e^{iSu}\,dS, \qquad \tau(S) = 1 - \frac{(|S|-1)_+}{\varepsilon},$$

so that our inner integral can be written

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(s(1+\varepsilon))\int_{-(1+\varepsilon)}^{(1+\varepsilon)} \tau(S)e^{iS(x-t)}\,ds\,dx$$
$$= \frac{1}{2\pi}\int_{-(1+\varepsilon)}^{(1+\varepsilon)} \tau(S)e^{-St}\int_{-\infty}^{\infty} g(x(1+\varepsilon))e^{iSx}\,dx\,dS$$

but this inner integral is always 0, by hypothesis. We have proven, then, that $\int_{-\infty}^{\infty} f(x)g(x(1+\varepsilon))\,dx = 0$, or $\int_{-\infty}^{\infty} f(x/(1+\varepsilon))g(x)\,dx = 0$. Letting $\varepsilon \to 0$ and using the dominated convergence theorem we obtain, finally $\int_{-\infty}^{\infty} f(x)g(x)\,dx = 0$ and our proof is complete.

Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122