PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 4, December 1982

NORM ATTAINING OPERATORS AND SIMULTANEOUSLY CONTINUOUS RETRACTIONS

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ABSTRACT. A compact metric space S is constructed and it is shown that there is a bounded linear operator $T: L^1[0,1] \to C(S)$ which cannot be approximated by a norm attaining operator. Also it is established that there does not exist a retract of $L^\infty[0,1]$ onto its unit ball which is simultaneously weak* continuous and norm uniformly continuous.

1. Introduction. Let X and Y be Banach spaces. A linear operator $T: X \to Y$ is called *norm attaining* if there is an $x \in X$ with $||x|| = 1$ and $||Tx|| = ||T||$. Schachermayer in [**5**] has constructed an example of an operator $T_0: L^1[0,1] \to C[0,1]$ such that if $||T - T_0|| \leq \frac{1}{2}$ and $||T|| \leq 1$ then T is not norm attaining. This is the first example of a pair of classical Banach spaces X, Y such that the norm attaining operators from X to Y are not dense. Unfortunately, Schachermayer's example is very lengthy and intricate.

In this note we provide a simple example of a compact metric space S and an operator $T_0: L^1[0,1] \to C(S)$ such that if $||T - T_0|| \leq \frac{1}{2}$ and $||T|| \leq 1$, then T is not norm attaining. This is Corollary 2. The ideas are definitely inspired by Schachermayer's construction.

This example will be a simple corollary of Theorem 1. Another corollary provides an answer to a question of Benyamini concerning retractions on dual spaces. By a selection argument there is always a weak* continuous retraction of the dual of any separable Banach space onto its unit ball (the retraction $x \to x/||x||$ for $||x|| \geq 1$ is norm continuous but never weak* continuous except in the finite dimensional case). Benyamini [**1**] proved an embedding result about separable $C(K)$ spaces by establishing the existence of a retraction of the dual of $C[0,1]$ onto its unit ball which is simultaneously weak* continuous and norm uniformly continuous. Then in [**2**] Benyamini goes on to show that such simultaneously continuous retractions exist in the dual of l^p or $L^p[0,1]$ for $1 < p < \infty$. We will show that no such retraction is possible in the dual of $L^1[0,1]$. This is Corollary 3.

Notation is standard (cf. [**3**]), all subsets of $[0,1]$ below are assumed to be Borel and m denotes Lebesgue measure.

Let $S_0 = \{\sum_{i=1}^n (1 - 2^{-i})\chi_{D_i} | D_i \subset [0,1]$ are disjoint and $mD_i < 2^{-1}\}$. Throughout this note S will denote the weak* closure of S_0 in $L^\infty[0,1] = L^1[0,1]^*$. Thus, S is a compact metric space whose essential property for our purposes is contained in the following.

Received by the editors February 2, 1982.
1980 *Mathematics Subject Classification*. Primary 46B25; Secondary 47A99.
[1]Research supported in part by NSF grant MCS8002544.

THEOREM 1. *Let $\psi\colon S \to L^\infty$ be weak* continuous and suppose, for each $s \in S$, $\|\psi(s) - s\|_\infty \le \frac{1}{2}$ and $\|\psi(s)\|_\infty \le 1$. Then for each $s \in S$ and each $k \ge 2$*

$$m\{\omega \in [0,1] | \psi(s)(\omega) \ge 1 - 2^{-k-2}\} \le 16 \cdot 2^{-k}.$$

The applications of this result stem from the following observation. Consider the points $g \in L^\infty[0,1]$ such that $\|g\| = 1$ and $m\{\omega : |g(\omega)| = \|g\| = 1\} > 0$. Such points g are dense in the surface of the ball in L^∞ and lie arbitrarily close to S, but they do not satisfy the distributional condition of Theorem 1. It is somewhat surprising that it is impossible to move the points in S around the unit ball of L^∞ in a weak* continuous manner (with no point moved more than norm distance $\frac{1}{2}$) by a map ψ such that one of the many nearby points g as above is in the range of ψ.

This theorem will be proved in §3. We now state the two applications which will be derived from Thoerem 1 in §2.

COROLLARY 2. *Define $T_0\colon L^1[0,1] \to C(S)$ by $T_0 f(s) = \int_0^1 f(\omega)s(\omega)\,d\omega$ for $f \in L^1$ and $s \in S$. If $T\colon L^1[0,1] \to C(S)$ is a linear operator with $\|T - T_0\| \le \frac{1}{2}$ and $\|T\| \le 1$ then T is not norm attaining.*

COROLLARY 3. *There is no weak * continuous and norm uniformly continuous retraction of $L^\infty[0,1]$ onto its unit ball.*

ACKNOWLEDGEMENT. The connection between a counterexample as in Corollary 2 and Corollary 3 was observed independently by the second author, W. Schachermayer and C. Stagall during a lecture by Y. Benyamini delivered at Oberwolfach in the summer of 1981.

2. Proofs of Corollaries 1 and 2. To see Theorem 1 implies Corollary 2 observe the well-known fact (cf. [3 and 4]) that $T\colon L^1[0,1] \to C(S)$ corresponds canonically to a weak* continuous function $\psi\colon S \to L^\infty[0,1]$ and that T is norm attaining if and only if there is a point $s \in S$ such that $\|\psi(s)\| = \|T\| = \|\psi\|$ and $\psi(s)$ is norm attaining as a functional on $L^1[0,1]$; i.e. there is a set E so that $mE > 0$ and $|\psi(s)| = \|\psi\|$ on E. Now, the identity map: $S \to L^\infty[0,1]$ represents T_0 and so $\|\psi(s) - s\| \le \frac{1}{2}$. Hence Theorem 1 implies the desired result.

Now, to see that Theorem 1 yields Corollary 3 suppose such a retraction φ exists. Choose $\delta > 0$ such that $\|f - g\| < \delta$ implies $\|\varphi(f) - \varphi(g)\| < \frac{1}{2}$. Let $s_0 \in S$ be of norm 1 (e.g. $s_0 = \sum_{n=1}^\infty (1 - 2^{-n})\chi_{D_n}$, $D_n = (2^{-n-1}, 2^{-n})$). There is a function $g \in L^\infty[0,1]$ with $g(\omega) = \|g\| = 1$ on a set of positive measure and $\|g - s_0\| < \delta$. Define $\psi\colon S \to L^\infty[0,1]$ by $\psi(s) = \varphi(s + g - s_0)$. Then $\|\psi(s) - s\| = \|\varphi(s + g - s_0) - \varphi(s)\| \le \frac{1}{2}$ since $\|s + g - s_0 - s\| < \delta$. Also $\|\psi\| \le 1$ and $\psi(s_0) = \varphi(g) = g$. This contradicts Theorem 1.

3. Proof of Theorem 1. In order to prove Theorem 1 we need the following:

LEMMA 1. *Suppose $s = \sum_{i=1}^n (1 - 2^{-i})\psi_{D_i} \in S_0$. Then $\|\psi(s)|_{D_i}\|_\infty \le 1 - 2^{-i-2}$ for each $i = 1, \ldots, n$.*

Before we prove Lemma 1, let us show how Theorem 1 follows from it.

To do this first observe that for $s = \sum_{i=1}^n (1 - 2^{-i})\psi_{D_i} \in S_0$ and for $k \le 2$,

$$m\{\omega | \psi(s)(\omega) \ge 1 - 2^{-k}\} \le 8 \cdot 2^{-k}. \tag{0}$$

For, if $A = \{\omega | \psi(s)(\omega) \geq 1 - 2^{-k}\}$, then by Lemma 1, $m(A \cap D_i) = 0$ when $1 - 2^{-i-2} < 1 - 2^{-k}$; i.e. when $i < k-2$. Also, $m(A \sim \bigcup_{i=1}^n D_i) = 0$ since $\|\psi(s) - s\| \leq \frac{1}{2}$ and $k \geq 2$. Thus $mA \leq \sum_{i=k-2}^n mD_i \leq \sum_{i=k-2}^\infty 2^{-i} = 8 \cdot 2^{-k}$, and (0) holds. Now if the conclusion of Theorem 1 fails, then for some $s \in S$ and $k \geq 2$, there is a set E with $mE = 16 \cdot 2^{-k}$ and $\int_E \psi(s) \geq mE(1 - 2^{-k-2})$. Given any $\epsilon > 0$, we may choose $s_1 \in S_0$ such that $\int_E \psi(s) \leq \int_E \psi(s_1) + \epsilon$, since S_0 is weak* dense in S. Using (0) we may break E into two sets E_1 and E_2 with $mE_1 = mE_2 = \frac{1}{2}mE = 8 \cdot 2^{-k}$ so that $\psi(s_1) < 1 - 2^{-k}$ on E_2. Thus

$$\begin{aligned} mE(1 - 2^{-k-2}) &\leq \int_E \psi(s) \leq \int_{E_1} \psi(s_1) + \int_{E_2} \psi(s_1) + \epsilon \\ &\leq mE_1 + (1 - 2^{-k}) mE_2 + \epsilon = mE(1 - 2^{-k-1}) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have a contradiction and Theorem 1 is proved.

In order to prove Lemma 1, we require the following simple observations:

Suppose $s = \sum_{i=1}^n \alpha_i \chi_{D_i}$, with $0 < \alpha_i < 1$ and the D_i's disjoint. Let A be a fixed subset of some D_i and let $\Pi = \{A_1, \ldots, A_k\}$ be a partition of A. Choose $C_j \subset A_j$ with $mC_j = 2\alpha_i mA_j/(i + \alpha_i)$ and define

$$T_\Pi s = \sum_{j=1}^k \frac{1 + \alpha_1}{2} \chi_{C_j} + s\chi_{\tilde{A}}.$$

The following are easy to check:

(1) $\int_{A_j} s = \int_{A_j} (1 + \alpha_i)\chi_{C_j}/2$.

(2) $\int_F s = \int_F T_\Pi s$ if $f \cap A = \varnothing$ or if F is a union of sets from Π. (This follows from (1).)

(3) $m(\{\omega | T_\Pi s(\omega) = 0\} \cap A) = (1 - \alpha_i)mA/(1 + \alpha_i)$.

(4) Weak*-$\lim_\Pi T_\Pi s = s$ where the limit is taken over partitions of A ordered by refinement.

Property (4) follows from (2). For, if $E \subset [0,1]$ and Π is a refinement of $\{A \cap E, A \sim E\}$, then

$$\begin{aligned} \int_E T_\Pi s &= \int_{E \sim A} T_\Pi s + \int_{E \cap A} T_\Pi s + \int_{A \sim E} T_\Pi s \\ &= \int_{E \sim A} s + \int_{E \cap A} s + \int_{A \sim E} s = \int_E s. \end{aligned}$$

Now, assume $s \in S_0$ and Lemma 1 fails. Then there is a set $A \subset D_i$ such that $mA > 0$, $(1/mA)\int_A \psi(s) > 1 - 2^{-i-2}$ and $mA + mD_{i+1} < 2^{-i-1}$. But if Π is a partition of A and if $E = \{\omega | T_\Pi s(\omega) = 0\} \cap A$, then $T_\Pi s \in S_0$ and

$$\int_A \psi(T_\Pi s) = \int_{A \sim E} \psi(T_\Pi s) + \int_E \psi(T_\Pi s) \leq m(A \sim E) + \frac{1}{2} mE.$$

A simple calculation using the definition of T_Π shows that $m(A \sim E) + \frac{1}{2}mE \leq mA(1 - 2^{-i-2})$.

Now, taking the weak* limit on Π and using (4) we get that

$$\int_A \psi(s) \leq mA(1 - 2^{-i-2}),$$

a contradiction.

This completes the proof of Lemma 1.

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