NORM ATTAINING OPERATORS AND SIMULTANEOUSLY CONTINUOUS RETRACTIONS

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ABSTRACT. A compact metric space S is constructed and it is shown that there is a bounded linear operator \( T: L^1[0,1] \to C(S) \) which cannot be approximated by a norm attaining operator. Also it is established that there does not exist a retract of \( L^\infty[0,1] \) onto its unit ball which is simultaneously weak* continuous and norm uniformly continuous.

1. Introduction. Let \( X \) and \( Y \) be Banach spaces. A linear operator \( T: X \to Y \) is called norm attaining if there is an \( x \in X \) with \( \|x\| = 1 \) and \( \|Tx\| = \|T\| \). Schachermayer in [5] has constructed an example of an operator \( T_0: L^1[0,1] \to C[0,1] \) such that if \( \|T - T_0\| < \frac{1}{2} \) and \( \|T\| \leq 1 \) then \( T \) is not norm attaining. This is the first example of a pair of classical Banach spaces \( X, Y \) such that the norm attaining operators from \( X \) to \( Y \) are not dense. Unfortunately, Schachermayer's example is very lengthy and intricate.

In this note we provide a simple example of a compact metric space \( S \) and an operator \( T_0: L^1[0,1] \to C(S) \) such that if \( \|T - T_0\| < \frac{1}{2} \) and \( \|T\| \leq 1 \), then \( T \) is not norm attaining. This is Corollary 2. The ideas are definitely inspired by Schachermayer's construction.

This example will be a simple corollary of Theorem 1. Another corollary provides an answer to a question of Benyamini concerning retractions on dual spaces. By a selection argument there is always a weak* continuous retraction of the dual of any separable Banach space onto its unit ball (the retraction \( x \to x/\|x\| \) for \( \|x\| \geq 1 \) is norm continuous but never weak* continuous except in the finite dimensional case). Benyamini [1] proved an embedding result about separable \( C(K) \) spaces by establishing the existence of a retraction of the dual of \( C[0,1] \) onto its unit ball which is simultaneously weak* continuous and norm uniformly continuous. Then in [2] Benyamini goes on to show that such simultaneously continuous retractions exist in the dual of \( L^p \) or \( Lp^*[0,1] \) for \( 1 < p < \infty \). We will show that no such retraction is possible in the dual of \( L^1[0,1] \). This is Corollary 3.

Notation is standard (cf. [3]), all subsets of \([0,1]\) below are assumed to be Borel and \( m \) denotes Lebesgue measure.

Let \( S_0 = \{\sum_{i=1}^{n} (1 - 2^{-i}) \chi_{D_i} : D_i \subset [0,1] \text{ are disjoint and } mD_i < 2^{-1}\} \). Throughout this note \( S \) will denote the weak* closure of \( S_0 \) in \( L^\infty[0,1] = L^1[0,1]^* \). Thus, \( S \) is a compact metric space whose essential property for our purposes is contained in the following.
THEOREM 1. Let $\psi : S \to L^\infty$ be weak$^*$ continuous and suppose, for each $s \in S$, $\|\psi(s) - s\|_\infty \leq \frac{1}{2}$ and $\|\psi(s)\|_\infty \leq 1$. Then for each $s \in S$ and each $k \geq 2$
\[ m\{\omega \in [0,1] | \psi(s)(\omega) > 1 - 2^{-k-2}\} \leq 16 \cdot 2^{-k}. \]

The applications of this result stem from the following observation. Consider the points $g \in L^\infty[0,1]$ such that $\|g\| = 1$ and $m\{\omega : |g(\omega)| = \|g\| = 1\} > 0$. Such points $g$ are dense in the surface of the ball in $L^\infty$ and lie arbitrarily close to $S$, but they do not satisfy the distributional condition of Theorem 1. It is somewhat surprising that it is impossible to move the points in $S$ around the unit ball of $L^\infty$ in a weak$^*$ continuous manner (with no point moved more than norm distance $\frac{1}{3}$) by a map $\psi$ such that one of the many nearby points $g$ as above is in the range of $\psi$.

This theorem will be proved in §3. We now state the two applications which will be derived from Theorem 1 in §2.

COROLLARY 2. Define $T_0 : L^1[0,1] \to C(S)$ by $T_0f(s) = \int_0^1 f(\omega)s(\omega) d\omega$ for $f \in L^1$ and $s \in S$. If $T : L^1[0,1] \to C(S)$ is a linear operator with $\|T - T_0\| \leq \frac{1}{2}$ and $\|T\| \leq 1$ then $T$ is not norm attaining.

COROLLARY 3. There is no weak$^*$ continuous and norm uniformly continuous retraction of $L^\infty[0,1]$ onto its unit ball.

ACKNOWLEDGEMENT. The connection between a counterexample as in Corollary 2 and Corollary 3 was observed independently by the second author, W. Schachermayer and C. Stagall during a lecture by Y. Benyamini delivered at Oberwolfach in the summer of 1981.

2. Proofs of Corollaries 1 and 2. To see Theorem 1 implies Corollary 2 observe the well-known fact (cf. [3 and 4]) that $T : L^1[0,1] \to C(S)$ corresponds canonically to a weak$^*$ continuous function $\psi : S \to L^\infty[0,1]$ and that $T$ is norm attaining if and only if there is a point $s \in S$ such that $\|\psi(s)\| = \|T\| = \|\psi\|$ and $\psi(s)$ is norm attaining as a functional on $L^1[0,1]$; i.e. there is a set $E$ so that $mE > 0$ and $|\psi(s)| = \|\psi\|$ on $E$. Now, the identity map: $S \to L^\infty[0,1]$ represents $T_0$ and so $\|\psi(s) - s\| \leq \frac{1}{3}$. Hence Theorem 1 implies the desired result.

Now, to see that Theorem 1 yields Corollary 3 suppose such a retraction $\varphi$ exists. Choose $\delta > 0$ such that $\|f - g\| < \delta$ implies $\|\varphi(f) - \varphi(g)\| < \frac{1}{2}$. Let $s_0 \in S$ be of norm 1 (e.g. $s_0 = \sum_{n=1}^{\infty} (1 - 2^{-n}) \chi_{D_n}$, $D_n = (2^{n-1}, 2^{-n})$). There is a function $g \in L^\infty[0,1]$ with $g(\omega) = \|g\| = 1$ on a set of positive measure and $\|g - s_0\| < \delta$. Define $\varphi : S \to L^\infty[0,1]$ by $\varphi(s) = \varphi(s + g - s_0)$. Then $\|\varphi(s) - s\| = \|\varphi(s + g - s_0) - \varphi(s)\| \leq \frac{1}{2}$ since $\|s + g - s_0 - s\| < \delta$. Also $\|\varphi\| \leq 1$ and $\varphi(s_0) = \varphi(g) = g$. This contradicts Theorem 1.

3. Proof of Theorem 1. In order to prove Theorem 1 we need the following:

LEMMA 1. Suppose $s = \sum_{i=1}^{n} (1 - 2^{-i})\psi_{D_i} \in S_0$. Then $\|\psi(s)\|_{D_i} \leq 1 - 2^{-i-2}$ for each $i = 1, \ldots, n$.

Before we prove Lemma 1, let us show how Theorem 1 follows from it.
To do this first observe that for $s = \sum_{i=1}^{n} (1 - 2^{-i})\psi_{D_i} \in S_0$ and for $k \leq 2$,
\[ m\{\omega | |\psi(s)(\omega) - 1| > 2^{-k}\} \leq 8 \cdot 2^{-k}. \]
For, if \( A = \{\omega | \psi(s)(\omega) \geq 1 - 2^{-k}\} \), then by Lemma 1, \( m(A \cap D_i) = 0 \) when \( 1 - 2^{-i-2} < 1 - 2^{-k} \); i.e. when \( i < k - 2 \). Also, \( m(\bigcup_{i=1}^{n} D_i) = 0 \) since \( ||\psi(s) - s|| \leq \frac{1}{2} \) and \( k \geq 2 \). Thus \( mA \leq \sum_{i=k-2}^{n} mD_i \leq \sum_{i=k-2}^{n} 2^{-i} = 8 \cdot 2^{-k} \), and (0) holds. Now if the conclusion of Theorem 1 fails, then for some \( s \in S \) and \( k \geq 2 \), there is a set \( E \) with \( mE = 16 \cdot 2^{-k} \) and \( \int_E \psi(s) \geq mE(1 - 2^{-k-2}) \). Given any \( \epsilon > 0 \), we may choose \( s_1 \in S_0 \) such that \( \int_E \psi(s) \leq \int_E \psi(s_1) + \epsilon \), since \( S_0 \) is weak* dense in \( S \). Using (0) we may break \( E \) into two sets \( E_1 \) and \( E_2 \) with \( mE_1 = mE_2 = \frac{1}{2} mE = 8 \cdot 2^{-k} \) so that \( \psi(s_1) < 1 - 2^{-k} \) on \( E_2 \). Thus

\[
mE(1 - 2^{-k-2}) \leq \int_E \psi(s) \leq \int_{E_1} \psi(s_1) + \int_{E_2} \psi(s_1) + \epsilon
\leq mE_1 + (1 - 2^{-k})mE_2 + \epsilon = mE(1 - 2^{-k-1}) + \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, we have a contradiction and Theorem 1 is proved.

In order to prove Lemma 1, we require the following simple observations:

Suppose \( s = \sum_{i=1}^{n} \alpha_i \chi_{D_i} \), with \( 0 < \alpha_i < 1 \) and the \( D_i \)'s disjoint. Let \( A \) be a fixed subset of some \( D_i \) and let \( \Pi = \{A_1, \ldots, A_k\} \) be a partition of \( A \). Choose \( C_j \subseteq A_j \) with \( mC_j = 2\alpha_i mA/(1 + \alpha_i) \) and define

\[
T_{\Pi}s = \sum_{j=1}^{k} \frac{1 + \alpha_i}{2} \chi_{C_j} + s \chi_{A}.
\]

The following are easy to check:

1. \( \int_{A_j} s = \int_{A_j} (1 + \alpha_i) \chi_{C_j}/2 \).
2. \( \int_{F} s = \int_{F} T_{\Pi}s \) if \( f \cap A = \emptyset \) or if \( F \) is a union of sets from \( \Pi \). (This follows from (1).)
3. \( m(\{\omega | T_{\Pi}s(\omega) = 0\} \cap A) = (1 - \alpha_i)mA/(1 + \alpha_i) \).
4. \( \text{Weak*}-\lim_{\Pi} T_{\Pi}s = s \) where the limit is taken over partitions of \( A \) ordered by refinement.

Property (4) follows from (2). For, if \( E \subset [0,1] \) and \( \Pi \) is a refinement of \( \{A \cap E, A \sim E\} \), then

\[
\int_E T_{\Pi}s = \int_{E \sim A} T_{\Pi}s + \int_{E \cap A} T_{\Pi}s + \int_{A \sim E} T_{\Pi}s
= \int_{E \sim A} s + \int_{E \cap A} s + \int_{A \sim E} s = \int_E s.
\]

Now, assume \( s \in S_0 \) and Lemma 1 fails. Then there is a set \( A \subset D_i \) such that \( mA > 0 \), \( (1/mA) \int_A \psi(s) > 1 - 2^{-i-2} \) and \( mA + mD_{i+1} < 2^{-i-1} \). But if \( \Pi \) is a partition of \( A \) and if \( E = \{\omega | T_{\Pi}s(\omega) = 0\} \cap A \), then \( T_{\Pi}s \in S_0 \) and

\[
\int_A \psi(T_{\Pi}s) = \int_{A \sim E} \psi(T_{\Pi}s) + \int_E \psi(T_{\Pi}s) \leq mA(A \sim E) + \frac{1}{2} mE.
\]

A simple calculation using the definition of \( T_{\Pi} \) shows that \( mA(A \sim E) + \frac{1}{2} mE \leq mA(1 - 2^{-i-2}) \).

Now, taking the weak* limit on \( \Pi \) and using (4) we get that

\[
\int_A \psi(s) \leq mA(1 - 2^{-i-2}),
\]

a contradiction.

This completes the proof of Lemma 1.
REFERENCES

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