NORM ATTAINING OPERATORS AND SIMULTANEOUSLY CONTINUOUS RETRACTIONS

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ABSTRACT. A compact metric space $S$ is constructed and it is shown that there is a bounded linear operator $T : L^1[0,1] \rightarrow C(S)$ which cannot be approximated by a norm attaining operator. Also it is established that there does not exist a retract of $L^\infty[0,1]$ onto its unit ball which is simultaneously weak* continuous and norm uniformly continuous.

1. Introduction. Let $X$ and $Y$ be Banach spaces. A linear operator $T : X \rightarrow Y$ is called norm attaining if there is an $x \in X$ with $||x|| = 1$ and $||Tx|| = ||T||$. Schachermayer in [5] has constructed an example of an operator $T_0 : L^1[0,1] \rightarrow C[0,1]$ such that if $||T - T_0|| \leq \frac{1}{2}$ and $||T|| \leq 1$ then $T$ is not norm attaining. This is the first example of a pair of classical Banach spaces $X$, $Y$ such that the norm attaining operators from $X$ to $Y$ are not dense. Unfortunately, Schachermayer’s example is very lengthy and intricate.

In this note we provide a simple example of a compact metric space $S$ and an operator $T : L^1[0,1] \rightarrow C(S)$ such that if $||T - T_0|| \leq \frac{1}{2}$ and $||T|| \leq 1$, then $T$ is not norm attaining. This is Corollary 2. The ideas are definitely inspired by Schachermayer’s construction.

This example will be a simple corollary of Theorem 1. Another corollary provides an answer to a question of Benyamini concerning retractions on dual spaces. By a selection argument there is always a weak* continuous retraction of the dual of any separable Banach space onto its unit ball (the retraction $x \rightarrow x/||x||$ for $||x|| \geq 1$ is norm continuous but never weak* continuous except in the finite dimensional case). Benyamini [1] proved an embedding result about separable $C(K)$ spaces by establishing the existence of a retraction of the dual of $C[0,1]$ onto its unit ball which is simultaneously weak* continuous and norm uniformly continuous. Then in [2] Benyamini goes on to show that such simultaneously continuous retractions exist in the dual of $L^p$ or $L^p[0,1]$ for $1 < p < \infty$. We will show that no such retraction is possible in the dual of $L^1[0,1]$. This is Corollary 3.

Notation is standard (cf. [3]), all subsets of $[0,1]$ below are assumed to be Borel and $m$ denotes Lebesgue measure.

Let $S_0 = \{\sum_{i=1}^{n}(1 - 2^{-i})x_{D_i} | D_i \subset [0,1], mD_i < 2^{-1}\}$. Throughout this note $S$ will denote the weak* closure of $S_0$ in $L^\infty[0,1] = L^1[0,1]^*$. Thus, $S$ is a compact metric space whose essential property for our purposes is contained in the following.

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Theorem 1. Let $\psi: S \to L^\infty$ be weak* continuous and suppose, for each $s \in S$, $\|\psi(s) - s\|_\infty \leq \frac{1}{2}$ and $\|\psi(s)\|_\infty \leq 1$. Then for each $s \in S$ and each $k \geq 2$

$$m\{\omega \in [0,1] | \psi(s)(\omega) \geq 1 - 2^{-k-2} \} \leq 16 \cdot 2^{-k}.$$ 

The applications of this result stem from the following observation. Consider the points $g \in L^\infty[0,1]$ such that $\|g\| = 1$ and $m\{\omega : |g(\omega)| = \|g\| = 1\} > 0$. Such points $g$ are dense in the surface of the ball in $L^\infty$ and lie arbitrarily close to $S$, but they do not satisfy the distributional condition of Theorem 1. It is somewhat surprising that it is impossible to move the points in $S$ around the unit ball of $L^\infty$ in a weak* continuous manner (with no point moved more than norm distance $\frac{1}{2}$) by a map $\psi$ such that one of the many nearby points $g$ as above is in the range of $\psi$.

This theorem will be proved in §3. We now state the two applications which will be derived from Theorem 1 in §2.

Corollary 2. Define $T_0: L^1[0,1] \to C(S)$ by $T_0f(s) = \int_0^1 f(\omega)s(\omega)d\omega$ for $f \in L^1$ and $s \in S$. If $T: L^1[0,1] \to C(S)$ is a linear operator with $\|T - T_0\| \leq \frac{1}{2}$ and $\|T\| \leq 1$ then $T$ is not norm attaining.

Corollary 3. There is no weak* continuous and norm uniformly continuous retraction of $L^\infty[0,1]$ onto its unit ball.

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2. Proofs of Corollaries 1 and 2. To see Theorem 1 implies Corollary 2 observe the well-known fact (cf. [3 and 4]) that $T: L^1[0,1] \to C(S)$ corresponds canonically to a weak* continuous function $\psi: S \to L^\infty[0,1]$ and that $T$ is norm attaining if and only if there is a point $s \in S$ such that $\|\psi(s)\| = \|T\| = \|\psi\|$ and $\psi(s)$ is norm attaining as a functional on $L^1[0,1]$; i.e. there is a set $E$ so that $mE > 0$ and $|\psi(s)| = \|\psi\|$ on $E$. Now, the identity map: $S \to L^\infty[0,1]$ represents $T_0$ and so $\|\psi(s) - s\| \leq \frac{1}{2}$. Hence Theorem 1 implies the desired result.

Now, to see that Theorem 1 yields Corollary 3 suppose such a retraction $\varphi$ exists. Choose $\delta > 0$ such that $\|f - g\| < \delta$ implies $\|\varphi(f) - \varphi(g)\| < \frac{1}{2}$. Let $s_0 \in S$ be of norm 1 (e.g. $s_0 = \sum_{n=1}^\infty (1 - 2^{-n}) \chi_{D_n}$, $D_n = (2^{-n-1}, 2^{-n})$). There is a function $g \in L^\infty[0,1]$ with $g(\omega) = \|g\| = 1$ on a set of positive measure and $\|g - s_0\| < \delta$. Define $\psi: S \to L^\infty[0,1]$ by $\psi(s) = \varphi(s + g - s_0)$. Then $\|\psi(s) - s\| = \|\varphi(s + g - s_0) - \varphi(s)\| \leq \frac{1}{2}$ since $\|s + g - s_0 - s\| < \delta$. Also $\|\psi\| \leq 1$ and $\psi(s_0) = \varphi(g) = g$. This contradicts Theorem 1.

3. Proof of Theorem 1. In order to prove Theorem 1 we need the following:

Lemma 1. Suppose $s = \sum_{i=1}^n (1 - 2^{-i}) \psi_{D_i} \in S_0$. Then $\|\psi(s)\|_{D_i} \leq 1 - 2^{-i-2}$ for each $i = 1, \ldots, n$.

Before we prove Lemma 1, let us show how Theorem 1 follows from it.

To do this first observe that for $s = \sum_{i=1}^n (1 - 2^{-i}) \psi_{D_i} \in S_0$ and for $k \leq 2$,

$$m\{\omega | \psi(s)(\omega) \geq 1 - 2^{-k} \} \leq 8 \cdot 2^{-k}.$$
For, if $A = \{\omega | \psi(s)(\omega) \geq 1 - 2^{-k}\}$, then by Lemma 1, $m(A \cap D_i) = 0$ when $1 - 2^{-i-2} < 1 - 2^{-k}$; i.e. when $i < k - 2$. Also, $m(A \sim \bigcup_{i=1}^{\infty} D_i) = 0$ since $||\psi(s) - s|| \leq \frac{1}{2}$ and $k \geq 2$. Thus $mA \leq \sum_{i=k-2}^{\infty} 2^{-i} = 8 \cdot 2^{-k}$, and (0) holds. Now if the conclusion of Theorem 1 fails, then for some $s \in S$ and $k \geq 2$, there is a set $E$ with $mE = 16 \cdot 2^{-k}$ and $\int E \psi(s) = mE(1 - 2^{-k-2})$. Given any $\epsilon > 0$, we may choose $s_1 \in S_0$ such that $\int E \psi(s) \leq \int E \psi(s_1) + \epsilon$, since $S_0$ is weak* dense in $S$. Using (0) we may break $E$ into two sets $E_1$ and $E_2$ with $mE_1 = mE_2 = \frac{1}{2}mE = 8 \cdot 2^{-k}$ so that $\psi(s_1) < 1 - 2^{-k}$ on $E_2$. Thus

$$mE(1 - 2^{-k-2}) \leq \int E \psi(s) \leq \int_{E_1} \psi(s_1) + \int_{E_2} \psi(s_1) + \epsilon$$

$$\leq mE_1 (1 - 2^{-k})mE_2 + \epsilon = mE(1 - 2^{-k-1}) + \epsilon.$$ 

Since $\epsilon > 0$ is arbitrary, we have a contradiction and Theorem 1 is proved.

In order to prove Lemma 1, we require the following simple observations:

Suppose $s = \sum_{i=1}^{n} \alpha_i \chi_{D_i}$, with $0 < \alpha_i < 1$ and the $D_i$'s disjoint. Let $A$ be a fixed subset of some $D_i$ and let $\Pi = \{A_1, \ldots, A_k\}$ be a partition of $A$. Choose $C_j \subset A_j$ with $mC_j = 2\alpha_i m_{A_j}/(1 + \alpha_i)$ and define

$$T_{\Pi}s = \sum_{j=1}^{k} \frac{1 + \alpha_i}{2} \chi_{C_j} + s\chi_A.$$

The following are easy to check:

(1) $\int_{A_j} s = \int_{A_j} (1 + \alpha_i) \chi_{C_j}/2.$

(2) $\int_F s = \int_F \psi s$ if $f \cap A = \emptyset$ or if $F$ is a union of sets from $\Pi$. (This follows from (1).)

(3) $m(\{\omega | T_{\Pi}s(\omega) = 0 \} \cap A) = (1 - \alpha_i)mA/(1 + \alpha_i)$.

(4) Weak*-$\lim_{\Pi} T_{\Pi}s = s$ where the limit is taken over partitions of $A$ ordered by refinement.

Property (4) follows from (2). For, if $E \subset [0,1]$ and $\Pi$ is a refinement of $\{A \cap E, A \sim E\}$, then

$$\int_{E} T_{\Pi}s = \int_{E \sim A} T_{\Pi}s + \int_{E \cap A} T_{\Pi}s + \int_{A \sim E} T_{\Pi}s$$

$$= \int_{E \sim A} s + \int_{E \cap A} s + \int_{A \sim E} s = \int_{E} s.$$ 

Now, assume $s \in S_0$ and Lemma 1 fails. Then there is a set $A \subset D_i$ such that $mA > 0$, $(1/mA) \int A \psi(s) > 1 - 2^{-i-2}$ and $mA + mD_{i+1} < 2^{-i-1}$. But if $\Pi$ is a partition of $A$ and if $E = \{\omega | T_{\Pi}s(\omega) = 0\} \cap A$, then $T_{\Pi}s \in S_0$ and

$$\int_{A} \psi(T_{\Pi}s) = \int_{E \sim A} \psi(T_{\Pi}s) + \int_{E} \psi(T_{\Pi}s) \leq mA(1 - E) + \frac{1}{2}mE.$$ 

A simple calculation using the definition of $T_{\Pi}$ shows that $mA(1 - E) + \frac{1}{2}mE \leq mA(1 - 2^{-i-1}).$

Now, taking the weak* limit on $\Pi$ and using (4) we get that

$$\int_A \psi(s) \leq mA(1 - 2^{-i-2}),$$

a contradiction.

This completes the proof of Lemma 1.
REFERENCES

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