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UNIQUENESS OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WHEN THE INITIAL SURFACE IS CHARACTERISTIC AT A POINT

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ABSTRACT. Uniqueness in the Cauchy problem for hyperbolic operators degenerate at a point on the initial surface depends on values of the coefficients of the lower order terms. If the operator P is doubly characteristic at the origin with respect to the $t = 0$ line, P has uniqueness for functions which are smooth enough if the coefficient of the D_t term does not lie in a certain discrete set of numbers.

Introduction. We consider operators of the form

$$P = A(x,t)D_t^2 + 2B(x,t)D_xD_t - C(x,t)D_x^2 + \lambda(x,t)D_t + R_1 \tag{1}$$

where all the coefficients are real C^∞ functions, A is nonnegative, C is positive, R_1 is a first order linear operator in D_x,

$$D_x = \partial/\partial x \quad \text{and} \quad D_t = \partial/\partial t.$$

When P is doubly characteristic at the origin with respect to the surface given by $t = 0$, uniqueness of the solution to the problem:

$$\begin{aligned} Pu &= F, \qquad (x,t) \in \Omega, \text{a neighborhood of the origin}, t \geq 0, \\ u &= u_0(x), \quad \text{if } t = 0, \\ D_t u &= u_1(x), \quad \text{if } t = 0, \end{aligned}$$

is shown to depend on the values of the lower order terms, principally λ. If λ lies outside a certain discrete set of values the solutions to the problem will be unique whenever μ is smooth enough.

If λ lies in this discrete set of values, uniqueness will depend on R_1. The dependence on the lower order terms is very complex. There is always uniqueness when

$$\lambda_0 = \lambda - D_xB - D_tA/2 \geq 0. \tag{2}$$

As λ_0 becomes more negative, u must be assumed to be smoother to get the same result.

There are some values of the coefficients for which no uniqueness results can be obtained. If $c > 0$,

$$Pu = x^2D_t^2u - c^2D_x^2u - (2n+1)cD_tu \tag{3}$$

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and

$$\tilde{X}u = xD_t u - cD_x u,$$

then

$$P\tilde{X}^N h(ct - x^2/2) = 0, \quad h \in C_c^\infty(R).$$

The assertion is proved by taking the support of h to be the positive halfline. Finally, if P is as in (3), $P+\gamma$ is shown to have uniqueness if γ is a nonzero function.

The operators. P is characteristic with respect to the initial surface at the origin when A vanishes at the origin, and doubly characteristic with respect to this surface when both B and grad A also vanish there.

In order to use the concatenation technique of Višik and Grušin, [**12**], it is necessary to factor the principal part of the operator in a neighborhood of the origin which we will take to be Ω. The following hypotheses are assumed:

(Ha) There is a C^∞ function, a, such that

$$A = a^2 \quad \text{and} \quad D_x a = a_x > 0.$$

(Hb) There is a C^∞ function, b, such that

$$B = ba.$$

Then

$$P = a^2 D_t^2 + 2abD_t D_x + b^2 D_x^2 - (C + b^2)D_x^2 + \lambda D_t + R_1.$$

Let

$$c^2 = C + b^2, \quad \text{where } c > 0.$$

Define

$$\tilde{Y} = aD_t + bD_x + cD_x \quad \text{and} \quad \tilde{X} = aD_t + bD_x - cD_x.$$

So

$$P = \tilde{Y}\tilde{X} + \lambda' D_t + R_2, \tag{4}$$

where R_2 is a first order operator in D_x and

$$\lambda' = \lambda - (b + c)a_x.$$

This form is used in the concatenations. The commutator, $[\tilde{Y}, \tilde{X}]$, of $\tilde{Y}$ and $\tilde{X}$ acting on a function is

$$[\tilde{Y}, \tilde{X}]u = (\tilde{Y}\tilde{X} - \tilde{X}\tilde{Y})u = (2a_x c D_t + R_3)u, \tag{5}$$

where R_3 is a first order operator in D_x. The hypotheses (1) and (Ha) guarantee $a_x c > 0$.

DEFINITION 1. The following norms are used:

$$\|u\|^2 = \|u\|_0^2 = \iint_\Omega |u|^2 \, dx \, dt, \quad u \in \mathcal{H}^2(\Omega).$$

Then

$$\|u\|_{N+1}^2 = \|D_t u\|_N^2 + \sigma^2 \|u\|_N^2 + \|D_x u\|_N^2.$$

The following norm is taken from [**1**]:

$$\|u\|_{1,1}^2 = \|aD_t u\|_0^2 + \sigma^2 \|au\|_0^2 + \|D_x u\|_0^2.$$

Note that for all u in $C_c^\infty(\Omega)$, and $\sigma > \sigma_0$:

$$||u||_{1,1} \leq ||u||_1.$$

Also, if $u \in C_c^\infty(\Omega)$,

$$||(D_t + \sigma)u||^2 = ||D_t u||^2 + \sigma^2 ||u||^2.$$

Similar statements are true when other norms are used.

DEFINITION 2. An operator, P, has $(UCP)_N$ in an open set, Ω, if, whenever u is an element of $H^N(\Omega)$, and supp $u \cap \{t \leq 0\}$ is compact, then $Pu = 0$ in Ω, implies $u = 0$ in a neighborhod of supp $u \cap \{t \leq 0\}$.

Let $\Omega^- = \Omega \cap \{t \leq 0\}$. The following lemma is only stated because its proof is an easy modification of that in [8].

LEMMA 1. *If there are constants c and σ_0 such that for every function v in $C_c^\infty(\Omega)$, and every $\sigma > \sigma_0$, the Carleman estimate*

$$C||e^{-\sigma t} Pv||_N \geq ||e^{-\sigma t} v||_0 \tag{6}$$

holds, then P has $(UCP)_{N+m}$. Here m is the order of P.

Define

$$P_\phi u = \exp(-\sigma t) P(\exp(\sigma t) u).$$

The substitution

$$v = \exp(\sigma t) u$$

shows that (6) is equivalent to the following: there is a constant c, such that

$$c||P_\phi u||_N \geq ||u||_0, \quad u \in C_c^\infty(\Omega), \sigma > \sigma_0.$$

This is the estimate used in proving the theorem.

Proof of the theorem when λ_0 is positive. Let A_t be the partial of A with respect to t, B_x, the partial of B with respect to x, and C_t and C_x be the partials of C with respect to t and x respectively.

LEMMA 2. *If P is as in* (1) *where A is nonnegative and C is positive, and*

$$0 < L < \lambda_0 = \lambda(0,0) - B_x(0,0) - A_t(0,0)/2,$$

then there are $c > 0$ and σ_0, depending only on L and the coefficients of P such that if Ω is a sufficiently small neighborhood of the origin,

$$\langle P_\phi u, (D_t + \sigma)u \rangle \geq c\sigma ||u||_{1,1}^2 + L||u||_1^2, \quad u \in C_c^\infty(\Omega), \sigma > \sigma_0.$$

PROOF. Integration by parts in D_t shows

$$\langle A(D_t + \sigma)^2 u, (D_t + \sigma)u \rangle = \sigma ||A^{1/2}(D_t + \sigma)u||^2 - \langle A_t/2(D_t + \sigma)u, (D_t + \sigma)u \rangle \tag{7}$$

whenever u is in $C_c^\infty(\Omega)$.

Integration by parts in D_x shows that if $u \in C_c^\infty(\Omega)$,

$$\langle 2BD_x(D_t + \sigma)u, (D_t + \sigma)u \rangle = -\langle B_x(D_t + \sigma)u, (D_t + \sigma)u \rangle. \tag{8}$$

An integration by parts in D_t followed by another in D_x shows if $u \in C_c^\infty(\Omega)$,

$$\begin{aligned} &-\langle CD_x^2 u, (D_t + \sigma)u \rangle \\ &\quad = \sigma ||C^{1/2} D_x u||^2 - \langle (C_t/2) D_x u, D_x u \rangle + \langle (C_x/2) D_x u, (D_t + \sigma)u \rangle. \end{aligned} \tag{9}$$

We use (1), (7), (8), and the last equation, (9), to get if $u \in C_c^\infty(\Omega)$,

$$\begin{aligned}\langle P_\phi u, (D_t+\sigma)u\rangle &= \sigma\|a(D_t+\sigma)u\|^2 + \|(\sigma C - C_x/2)^{1/2} D_x u\|^2 \\ &\quad + \|(\lambda_0)^{1/2}(D_t+\sigma)u\|^2 + \langle R_4 u, (D_t+\sigma)u\rangle\end{aligned} \tag{10}$$

where R_4 is a first order operator in D_x.

Schwarz's inequality implies that for some R and any positive ϵ,

$$\langle R_4 u, (D_t+\sigma)u\rangle \le \epsilon\|(D_t+\sigma)u\|^2 + 1/(2\epsilon)\|Ru\|^2, \quad u \in C_c^\infty(\Omega).$$

Fix ϵ so that $L = \lambda_0 - \epsilon > 0$ after shrinking Ω if necessary. Then choose σ_0 so that $\sigma_0 C - C_x/2 - R/2\epsilon)$ is positive. This implies that the first three terms on the right in (10) dominate the last term on the right, which proves the lemma. □

COROLLARY. *If P is as in* (1) *and* (Ha) *and* (Hb) *hold, the conclusion of Lemma 2 is true if*

$$\lambda - a_x b > L > 0.$$

PROOF. Both ab_x and $A_t/2 = aa_t$ vanish at the origin.

Proof of the theorem when λ_0 is negative. It is easy to show that $P_\phi u$ dominates $Y^N u$ if λ_0 is not less than $-2Na_x c$, where

$$Yu = e^{-\sigma t}\tilde{Y}(e^{\sigma t}u), \quad u \in C_c^\infty(\Omega),$$

and

$$Y^{N+1}u = YY^N u,$$

But no inequality which is uniform in σ bounds Yu away from 0 in a neighborhood of the origin. To prove the theorem when $\lambda_0 < 0$, more structure is necessary. The next lemma handles the commutators between Y^N and P_ϕ. Let

$$Xu = e^{-\sigma t}X(e^{\sigma t}u), \quad u \in C_c^\infty(\Omega).$$

LEMMA 3. *If P is as in* (1) *and* (Ha) *and* (Hb) *are satisfied, then*

$$Y^N P_\phi u = (P_\phi + N[Y,X])Y^N u + \sum_{k<N} L_k Y^k u, \quad u \in C_c^\infty(\Omega). \tag{11}$$

Here L_k is a first order operator in D_x and D_t.

PROOF. The proof is by induction. The lemma is clearly true when $N = 0$. The induction hypothesis is (11). Then if $u \in C_c^\infty(\Omega)$,

$$Y^{N+1}P_\phi u = Y\left[(XY + \lambda'(D_t+\sigma) + R_2 + N[Y,X])Y^N u + \sum_{k<N} L_k Y^k u\right].$$

Y is commuted with the rest of the right-hand side above. The commutator satisfies

$$[Y, XY] = [Y,X]Y.$$

All the other commutators are first order. This recovers the inductive hypothesis. The lemma follows. □

COROLLARY. *If P is as in* (1) *and satisfies* (Ha) *and* (Hb), *and if*

$$\lambda(0,0) - a_x(0,0)b(0,0) = \lambda_0(0,0) > -2Na_x(0,0)c(0,0)$$

and

$$P_\phi^N = P_\phi + N[Y,X]$$

then there is some neighborhood Ω of the origin where P^N satisfies the hypotheses of Lemma 2.

PROOF. Define

$$\lambda_N = \lambda_0 + 2Na_x c \tag{12}$$

then the hypothesis of the corollary implies that in some neighborhood of the origin λ_N is strictly positive, and must be bounded away from 0 in some neighborhood of the origin which is taken to be Ω. But

$$P_\phi^N = XY + \lambda_N D_t + R_N$$

where R_N is a first order operator in D_x. This proves the corollary. □

Now the main theorem will be proved.

THEOREM 1. *If P is as in* (1) *and* (Ha) *and* (Hb) *are satisfied then P has $(UCP)_{N+2}$ if*

$$\lambda' + 2ka_x(0,0)c(0,0) \neq 0 \ \textit{for any } k = 1,2,3,\ldots \tag{h1}$$

and

$$\lambda_0 + 2Na_x(0,0)c(0,0) > 0. \tag{h2}$$

PROOF. Lemma 3 implies that there are constants c, r, and σ_0 such that if $u \in C_c^\infty(\Omega)$ and $\sigma > \sigma_0$,

$$c||P_\phi u||_N \geq ||Y^N P_\phi u||_0 \geq ||P_\phi^N Y^N u||_0 - S_0. \tag{13}$$

Here

$$S_J = \sigma^J \sum_{k<N-J} r_k ||Y^k u||_1.$$

The corollary to Lemma 3 and (h2) establish that P^N satisfies the hypothesis of Lemma 2, so there are constants c and σ_0, such that if $u \in C_c^\infty(\Omega)$ and $\sigma > \sigma_0$,

$$c\langle P_\phi^N Y^N u, (D_t + \sigma)Y^N u\rangle \geq \sigma ||Y^N u||_{1,1}^2 + ||Y^N u||_1^2. \tag{14}$$

Since

$$||(D_t + \sigma)v||_0 \leq ||v||_1$$

and because the 1,1 norm is smaller than the 1 norm, (14) implies there are constants, c and σ_0, such that if $u \in C_c^\infty(\Omega)$, $\sigma > \sigma_0$ and $N \geq 1$,

$$c||P_\phi u||_N \geq \sigma ||Y^N u||_{1,1} + ||Y^N u||_1.$$

Substitution of the last inequality into (13) implies there is a constant c such that if $u \in C_c^\infty(\Omega)$ and $\sigma > \sigma_0$,

$$c||P_\phi u||_N \geq \sigma ||Y^N||_{1,1} + ||Y^N u||_1 - S_0.$$

This inequality is the first step in an inductive argument. The inductive hypothesis is that there are constants c, r, and σ_0, depending on J, N, and P, but not on $u \in C_c^\infty(\Omega)$ or $\sigma > \sigma_0$, such that

$$c||P_\phi u||_N \geq \sigma^J(\sigma ||Y^{N-J} u||_{1,1} + ||Y^{N-J} u||_1) - S_J.$$

We now try to recover the inductive hypothesis for $J+1$. The inductive hypothesis for J implies there are constants c', r', and σ_0 such that if $u \in C_c^\infty(\Omega)$ and $\sigma > \sigma_0$,

$$c'||P_\phi u||_N \geq \sigma^{J+1} ||XYY^{N-J-1} u||_0 - S_J. \tag{16}$$

But the definition of the norms implies there is a constant c'' such that

$$c''\|P_\phi u\|_N \geq \sigma^{J+1}\|Y^{N-J-1}P_\phi u\|_0, \quad u \in C_c^\infty(\Omega),\ \sigma > \sigma_0.$$

Lemma 3 applied to the right-hand side of the inequality above implies the existence of constants c''', r'', and σ_0 such that if $u \in C_c^\infty(\Omega)$ and $\sigma > \sigma_0$,

$$c'''\|P_\phi u\|_N \geq \sigma^{J+1}\|P_\phi^{N-J-1}Y^{N-J-1}u\|_0 - S_{J+1}. \tag{17}$$

Subtracting (16) from the last inequality gives the following for $u \in C_c^\infty(\Omega)$ and $\sigma > \sigma_0$: if $\underline{c} = c' + c'''$, and λ_{N-J-1} is as in (12),

$$\underline{c}\|P_\phi u\|_N \geq \sigma^{J+1}\|\lambda_{N-J-1}(D_t+\sigma)Y^{N-J-1}u\|_0 - \sigma^J r'\|Y^{N-J-1}u\|_1 - S_{J+1}.$$

Here r'' and σ_0 have been increased to absorb all the terms of S_J but the $(N-J-1)$st. The $(N-J-1)$st term is the one which must be dominated to recover the inductive hypothesis.

Since (h2) guarantees that $\lambda_{N-J-1} = 0$ in Ω, we can add a multiple, m, of the last inequality to (17) to get for each $u \in C_c^\infty(\Omega)$ and $\sigma > \sigma_0$,

$$\underline{c}'\|P_\phi u\|_N \geq \sigma^{J+1}\|P_\phi^g Y^{N-J-1}u\|_0 - \sigma^J m r'\|Y^{N-J-1}u\|_1 - S_{N+1},$$

where $\underline{c}' = m\underline{c} + c'''$. The multiple m is chosen so that P_ϕ^g satisfies the hypothesis of Lemma 2. This now implies that for some new $\underline{c}''$ and σ_0, if $u \in C_c^\infty(\Omega)$ and $\sigma > \sigma_0$,

$$\begin{aligned}\underline{c}''\|P_\phi u\|_N \geq \sigma^{J+1}&(\sigma\|Y^{N-J-1}u\|_{1,1} + \|Y^{N-J-1}u\|_1)\\ &- \sigma^J r\|Y^{N-J-1}u\|_1 - S_{N+1}.\end{aligned}$$

If σ_0 is taken bigger than r, then the inductive hypothesis is recovered. This proves the estimate. The theorem follows from Lemma 1. □

With the introduction of more machinery, as in [**1** and **2**], a slightly stronger theorem can be proved. Is is possible to show that the conditions of Theorem 1 imply P has $(UCP)_{N+1}$. It is probably not possible to conclude that P has uniqueness for functions with continuous second partials when λ_0 is negative. It is also possible to strengthen the conclusion of Theorem 2 slightly, but the gain is small compared to the increase in difficulty of the proof.

COROLLARY. *If P is as in* (1) *and* (Ha) *and* (Hb) *are satisfied, then P has $(UCP)_{N+2}$ if*

$$\lambda \neq (b-(2k-1)c)a_x, \quad k = 1,2,3,\ldots, \tag{h1$'$}$$

and

$$\lambda > (b-2Nc)a_x. \tag{h2$'$}$$

PROOF. The proof that (h2)$'$ is equivalent to (h2) is exactly as in the corollary to Lemma 2. The equivalence of (h1)$'$ to (h1) follows directly from (2), the definition of λ_0.

COROLLARY 2. *If P is as in* (1) *and* (Ha) *and* (Hb) *hold, then P has $(UCP)_{N+2}$ if for every positive integer k, when $t = 0$,*

$$\lambda(0,0) \neq \lim_{x\to 0}\{A_x[B-(2k-1)(AC+B^2)^{1/2}]/(2A)\} \tag{hl$''$}$$

and when $t = 0$,

$$\lambda(0,0) > \lim_{x\to 0}\{A_x[B-2N(AC+B^2)^{1/2}]/(2A)\}. \tag{h2$''$}$$

PROOF. This follows directly from the definitions of a, b, and c. We have

$$a_x = A_x/(2A^{1/2}) \quad \text{and} \quad ba_x = A_x B/(2A).$$

Also

$$ca_x = A_x(AC + B^2)^{1/2}/(2A). \quad \square$$

The final theorem shows the complex dependence of uniqueness on the terms R_N when λ violates (h1).

THEOREM 2. *Let γ be a nonvanishing function,*

$$Pu = (XY + N[Y,X] + \gamma)u,$$

$$Y\gamma = 0 \quad \textit{and} \quad [Y,[Y,X]] = 0.$$

Then P has $(UCP)_{N+4}$.

PROOF. Application of Lemma 2 shows there are constants c and σ_0, depending on P but not on $u \in C_c^\infty(\Omega)$ or $\sigma > \sigma_0$, such that

$$c||P_\phi u||_{N+2} \geq ||Y^{N+1}P_\phi u||_1 = ||P_\phi^{N+1}Y^{N+1}u||_1.$$

Here P^{N+1} satisfies the hypothesis of Lemma 2. If for $u \in C_c^\infty(\Omega)$,

$$||u||_{2,1}^2 = ||u_t||_{1,1}^2 + ||u_x||_{1,1}^2 + \sigma^2||u||_{1,1}^2,$$

an easy extension of Lemma 2 shows there are constants c and σ_0 such that

$$c||P_\phi u||_{N+2} \geq \sigma||Y^{N+1}u||_{2,1} + ||Y^{N+1}u||_2, \quad u \in C_c^\infty(\Omega),\ \sigma > \sigma_0.$$

This implies there are c' and σ_0 such that if $u \in C_c^\infty(\Omega)$ and $\sigma > \sigma_0$,

$$c'||P_\phi u||_{N+2} \geq \sigma||XYY^N u||_1.$$

But the definitions of the norms imply that for some c and σ_0, if $u \in C_c^\infty(\Omega)$ and $\sigma > \sigma_0$

$$c''||P_\phi u||_{N+2} \geq \sigma||Y^N P_\phi u||_1 = \sigma||P_\phi^N Y^N u||_1. \tag{18}$$

Here a whole derivative has been lost. Subtracting (11) from (18) implies that if $u \in C_c^\infty(\Omega)$ and $\sigma > \sigma_0$,

$$(c'' + c')||P_\phi u||_{N+2} \geq \sigma|\gamma(0,0)/2|\,||Y^N u||_1.$$

The induction starts here. The inductive hypothesis is: there are c and σ_0 such that if $u \in C_c^\infty(\Omega)$ and $\sigma > \sigma_0$,

$$c||P_\phi u||_{N+2} \geq \sigma^{J+1}||Y^{N-J}u||_1.$$

The induction is similar to that in Theorem 1. The hypotheses of this theorem guarantee all the commutators except $[Y,X]$ vanish. When $J = N$, an estimate which proves the theorem is obtained. $\square$

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