ORTHOGONAL EXPANSIONS OF VECTORS
IN A HILBERT SPACE FOR NON-GAUSSIAN MEASURES

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Abstract. Let $\mathcal{G}$ be a separable Hilbert space and $\mu$ a probability Radon measure on $\mathcal{G}$ of second order. Then there exist $(a_n) \in l^2$, an O.N.S. $(x_n) \subset \mathcal{G}$ and an O.N.S. $(\xi_n) \subset H$ such that the orthogonal series $\sum_{n=1}^{\infty} a_n \xi_n(x) x_n$ converges in $\mathcal{G}$ $\mu$-almost everywhere and it holds that $x = \sum_{n=1}^{\infty} a_n \xi_n(x) x_n$, $\mu$-almost everywhere, where $H$ is the generating Hilbert space of $\mu$. In the case where $\mu$ is a Gaussian measure, a similar result was proved by Kuelbs [2] in general Banach spaces.

1. Let $\mathcal{K}$ be a separable (real or complex) Hilbert space and $\mu$ a probability measure on $\mathcal{K}$ of second order, that is, $\int_{\mathcal{K}} \|x\|^2 \, d\mu(x) < +\infty$. $\mathcal{K}'$ denotes the dual space of $\mathcal{K}$ and is identified with $\mathcal{K}$ by the conjugate linear isometry $\mathcal{K} \ni x \mapsto (|x|) \in \mathcal{K}'$, where $(|)$ is the inner product of $\mathcal{K}$ (Riesz's theorem).

Let $R: \mathcal{K}' \to L^2(\mu)$ be $R(y) = (|y|)$. We let $H$ be the $L^2(\mu)$-closure of $R\mathcal{K}'$. Then $R: \mathcal{K}' \to H$ is a continuous (conjugate) linear mapping since

$$\|R(y)\|^2 = \int (|x|y)^2 \, d\mu(x) \leq \left( \int \|x\|^2 \, d\mu(x) \right) \|y\|^2.$$

Theorem. There exist $(a_n) \in l^2$, an O.N.S. $(x_n) \subset \mathcal{K}$ and an O.N.S. $(\xi_n) \subset H$ such that

$$x = \sum_{n=1}^{\infty} a_n \xi_n(x) x_n, \quad \mu\text{-almost everywhere},$$

where the right-hand side converges in $\mathcal{K}$ for $\mu$-almost every $x$.

Proof. First we show that $R: \mathcal{K}' \to H$ is a Hilbert-Schmidt operator. Let $(e_n)$ be a C.O.N.S. in $\mathcal{K}'$, then we have

$$\sum_{n=1}^{\infty} \|R(e_n)\|^2 = \sum_{n=1}^{\infty} \int (|x|e_n)^2 \, d\mu(x) = \int \|x\|^2 \, d\mu(x) < +\infty,$$

which proves the assertion.

By the well-known representation theorem of a Hilbert-Schmidt operator, there exist $(a_n) \in l^2$, an O.N.S. $(x_n) \subset \mathcal{K}$ and an O.N.S. $(\xi_n) \subset H$ such that

$$R(y)(x) = \sum_{n=1}^{\infty} a_n (x_n|y) \xi_n(x) \quad \text{in } H \subset L^2(\mu).$$
We prove that the orthogonal series \( \sum_{n=1}^{\infty} a_n \xi_n(x) x_n \) converges in \( \mathcal{C} \) \( \mu \)-almost everywhere. Since
\[
\left\| \sum_{n=k}^{m} a_n \xi_n(x) x_n \right\|^2 = \sum_{n=k}^{m} a_n^2 | \xi_n(x) |^2,
\]
it is sufficient to show the series \( \sum_{n=k}^{m} a_n^2 | \xi_n(x) |^2 \) converges \( \mu \)-almost everywhere. By the equality
\[
\int \sum_{n=k}^{m} a_n^2 | \xi_n(x) |^2 d\mu(x) = \sum_{n=k}^{m} a_n^2
\]
and by \( (a_n) \in l^2 \), it follows that the sequence \( \sum_{n=1}^{k} a_n^2 | \xi_n(x) |^2 \), \( k = 1, 2, \ldots \), converges in \( L^2(\mu) \). Hence there is a subsequence which converges \( \mu \)-almost everywhere. But the sequence is nonnegative and nondecreasing, so the sequence itself converges \( \mu \)-almost everywhere.

Lastly we show that \( x - \sum_{n=1}^{\infty} a_n \xi_n(x) x_n \), \( \mu \)-almost everywhere. Let \( (e_k) \) be a C.O.N.S. in \( \mathcal{C}' \). Then we have for \( \mu \)-almost every \( x \),
\[
(x | e_k) = R(e_k)(x) = \sum_{n=1}^{\infty} a_n (x_n | e_k) \xi_n(x)
\]
\[
= \left( \sum_{n=1}^{\infty} a_n \xi_n(x) x_n | e_k \right), \quad k = 1, 2, \ldots.
\]
Since \( (e_k) \) separates \( \mathcal{C} \), we get the assertion.

This completes the proof.

2. The Hilbert space \( H \) in the preceding section coincides with the generating Hilbert space of \( \mu \) constructed by Kuelbs [3, Lemma 2.1]. So by the representation of the mapping \( R \) obtained by Kuelbs [3], we can determine \( (a_n), (x_n) \) and \( (\xi_n) \) explicitly as follows. Let \( R^* : H' \to \mathcal{C} \) be the adjoint linear operator and set \( T = R \circ R^* : H' = H \to \mathcal{C} = \mathcal{C}' \to H \). Then it holds that
\[
R^*(\xi) = \int x \xi(x) \ d\mu(x)
\]
(Bochner integral, Kuelbs [3, (2.1)]),
\[
T(\xi)(x) = \int (x | y) \overline{\xi(y)} \ d\mu(y), \quad \xi \in H
\]
(Kuelbs [3, (2.2)]).

Then \( T : H \to H \) (or \( L^2(\mu) \to L^2(\mu) \)) is a positive selfadjoint nuclear linear operator (in fact, \( T \) is an integral operator with the integrable kernel \( (x | y) \), \( x, y \in \mathcal{C} \)). Let \( c_n \geq 0, \sum_{n=1}^{\infty} c_n < +\infty \) be the eigenvalues and \( (\xi_n) \subset H \) the eigenfunctions of \( T \). Put \( x_n = c_n^{-1/2} R^*(\xi_n) \). Then we have
\[
(x_n | x_m) = c_n^{-1/2} c_m^{-1/2} \left( \int x \xi_n(x) \ d\mu(x) | \int y \xi_m(y) \ d\mu(y) \right)
\]
\[
= c_n^{-1/2} c_m^{-1/2} \left[ \int (x | y) \overline{\xi_m(y)} \ d\mu(y) \right] \xi_n(x) \ d\mu(x)
\]
\[
= c_n^{-1/2} c_m^{-1/2} c_m \int \xi_m(x) \xi_n(x) \ d\mu(x) = 8_{n,m}.
\]
So we get \( x = \sum c_n^{1/2} \xi_n(x)x_n \) \( \mu \)-almost everywhere. Note that this procedure is very similar to Kuelbs [2, §4].

3. Let \( \nu \) be the Gaussian Radon measure on \( \mathcal{K} \) with the characteristic functional \( \exp(-\int |\langle x, x'\rangle|^2 d\mu(x)/2), x' \in \mathcal{K}' \). Then by Kuelbs [2, Theorem 3.1], it follows that \( x = \sum a_n \xi_n(x)x_n \) \( \nu \)-almost everywhere. Kuelbs' arguments are as follows. For the Gaussian measure \( \nu \), the orthogonality of \( (\xi_n) \) in \( L^2(\nu) \) is equivalent to the independence of \( (\xi_n) \) as random variables on \( (\mathcal{K}, \nu) \). And by the convergence theorem of sums of independent random variables due to Ito and Nisio [1], it follows that \( x = \sum a_n \xi_n(x)x_n \) \( \nu \)-almost everywhere. The purpose of this note is to show \( x = \sum a_n \xi_n(x)x_n \) for \( \mu \)-almost every \( x \). Note that \( \nu(A) = 1 \) does not imply \( \mu(A) = 1 \) in general. Note also that the orthogonal random variables \( (\xi_n) \) are not independent over \( (\mathcal{K}, \mu) \), so we cannot use Ito and Nisio [1]. To prove our theorem the Hilbert-Schmidtness of the operator \( R \) is essential. Our result does not hold for general Banach space \( \mathcal{K} \).

REFERENCES


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