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A SHORT PROOF THAT COMPACT QUASIDEVELOPABLE SPACES ARE METRIZABLE

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ABSTRACT. A new characterization of quasidevelopable spaces is given that allows an easier proof that compact quasidevelopable spaces are metrizable.

In [$\mathbf{B}_1$, $\mathbf{B}_2$] it was shown that a compact quasidevelopable Hausdorff space is metrizable. In this note a new characterization of quasidevelopable spaces is given that allows a much easier proof. Let all spaces be at least Hausdorff spaces and let N denote the set of natural numbers.

A space X is *quasidevelopable* if there exists a sequence $G_1, G_2, \ldots$ (called the *quasidevelopment*) of collections of open subsets of X such that if x is in an open set U, then there exists $n \in N$ such that $\varnothing \neq \mathrm{st}(x, G_n) \subset U$.

THEOREM 1. *A space (X, τ) is quasidevelopable if and only if there exists a function $g\colon N \times X \to \tau$ (g is called q.d. function) such that*

(i) *if $x \in X$, then $\{g(n, x) \mid n \in N, g(n, x) \neq \varnothing\}$ is a nonincreasing local base at x,*

(ii) *if $x \in g(n, y)$, then $g(n, x) \neq \varnothing$, and*

(iii) *if $\{x\}$ is not open in X and if $g(n, z_n)$ contains x and y_n for each $n \in N$ such that $g(n, x) \neq \varnothing$, then some subsequence of $\langle y_n \mid g(n, x) \neq \varnothing \rangle$ converges to x.*

PROOF. Let $G_1, G_2, \ldots$ be a quasidevelopment for X. If $x \in X$ and $x \in \bigcup G_n$, let $g'(n, x)$ be any element of G_n that contains x. Let

$$g(n, x) = \bigcap \{g'(i, x) \mid i \leqslant n, x \in \bigcup G_i\}.$$

Clearly (i) and (ii) are satisfied. If $\{x\}$ is not open in X and x and y_n are in $g(n, z_n)$, then $y_n \in \mathrm{st}(x, C_n)$ for all $n \in N$ such that $x \in \bigcup G_n$. Thus some subsequence of $\langle y_n \mid g(n, x) \neq \varnothing \rangle$ converges to x.

If $g\colon N \times X \to \tau$ is a q.d. function, let $G_1 = \{\{x\} \mid \{x\} \text{ open in } X\}$ and $G_{n+1} = \{g(n, x) \mid x \in X, g(n, x) \neq \varnothing\}$. It easily follows that $G_1, G_2, \ldots$ is a quasidevelopment for X. □

This characterization should be compared with a similar characterization of Moore spaces in [H].

THEOREM 2. *Let M be an infinite subset of N and g a q.d. function on X. If $g(m, y_m) \supset g(m+1, y_{m+1})$ for each $m \in M$ and the sequence $\langle y_m \mid m \in M \rangle$ converges to some $p \in \bigcap \{g(m, y_m) \mid m \in M\}$, then $\{g(m, y_m) \mid m \in M\}$ is a local base at p.*

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PROOF. Let n_1 be the first natural number such that $g(n_1, p) \neq \varnothing$. Let $m_1 \geqslant n_1$ be the first member of M such that $y_{m_1} \in g(n_1, p)$. Then $g(n_1, y_{m_1}) \neq \varnothing$ and $g(n_1, y_{m_1}) \supseteq g(m_1, y_{m_1})$. Inductively let n_k be the first natural number larger than n_{k-1} such that $g(n_k, p) \neq \varnothing$. Let m_k be the first natural number such that $m_k \geqslant n_k$ and $m_k > m_{k-1}$ such that $y_{m_k} \in g(n_k, p)$. Thus $g(n_k, y_{m_k}) \neq \varnothing$ and $g(n_k, y_{m_k}) \supseteq g(m_k, y_{m_k})$.

If $\{g(n_k, y_{m_k}) \mid k \in N\}$ is not a local base at p then there is an open set U containing p such that for each $k \in N$ there exists $t_k \in g(n_k, y_{m_k})$ such that $t_k \notin U$. But, by (iii) of Theorem 1, some subsequence of $\langle t_k \mid k \in N \rangle$ converges to p. From this contradiction it follows that $\{g(n_k, y_{m_k}) \mid k \in N\}$ is a local base at p. Since $g(n_k, y_{m_k}) \supset g(m_k, y_{m_k})$, it follows that $\{g(m, y_m) \mid m \in M\}$ is a local base at p. □

Recall that a perfect (= closed sets are G_δ-sets) quasidevelopable space is developable [$\mathbf{B_2}$].

THEOREM 3. *A compact quasidevelopable space is metrizable.*

PROOF. Let X be a compact quasidevelopable space with q.d. function g and let $D = \{x \mid \{x\} \text{ is open in } X\}$. Let C be a closed set in $X - D$. Then C is compact. If $x \in C$, let $n(x, 1)$ be the first natural number such that $g(n(x, 1), x) \neq \varnothing$. Let H_1 be a finite subcollection of $\{g(n(x, 1), x) \mid x \in C\}$ which covers C. Suppose $H_1, \ldots, H_{k-1}$ have been chosen such that each H_i, $1 < i \leqslant k - 1$, is a finite open cover of C and if $h \in H_{i+1}$, then there exists $h' \in H_i$ such that $h' \supset \mathrm{cl}(h)$ (= closure of h in X). If $x \in C$, let $n(x, k) > k$ be the first natural number such that $\mathrm{cl}(g(n(x, k), x)) \subset \bigcap \{h \in H_{k-1} \mid x \in h\}$. Let H_k be a finite subcollection of $\{g(n(x, k), x) \mid x \in C\}$ which covers C.

Suppose there exists $q \in X - C$ such that $q \in \bigcup H_k$ for each $k \in N$. Then, using König's lemma [**K**], there is a sequence $\langle h_k \mid k \in N \rangle$ such that $h_k \in H_k$, $\mathrm{cl}(h_{k+1}) \subset h_k$ and $q \in h_k$ for each $k \in N$. For each $k \in N$, $h_k = g(n_k, x_k)$ for some $x_k \in C$ where we have written $n_k = n(x_k, k)$. Since $x_k \in C$ and since X is first-countable, the sequence $\langle x_k \mid k \in N \rangle$ has a subsequence which converges to some $p \in \bigcap \{\mathrm{cl}(h_k) \mid k \in N\} \cap C$. Without loss of generality let $\langle x_k \mid k \in N \rangle$ converge to p. By Theorem 2, $\{g(n_k, x_k) \mid k \in N\}$ is a local base at p. Thus, for some $k \in N$, $g(n_k, x_k) \subset X - \{q\}$. From this contradiction it follows that $q \notin \bigcup H_k$ and, hence, $C = \bigcap \{\bigcup H_i \mid i \in N\}$. Thus C is a G_δ-set in X.

If K is a closed subset of X then $K = C \cup D'$ where $D' \subset D$ and $C \cap D = \varnothing$. Since $C \subset X - D$, $C = \bigcap \{O_i \mid i \in N\}$ where each O_i is open in X. Then $K = \bigcap \{O_i \cup D' \mid i \in N\}$ and thus, X is perfect. Thus X is a compact developable space and, hence, metrizable [$\mathbf{B_2}$]. □

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