

ORDER AND DIMENSION

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ABSTRACT. It will be shown that the conjectured Jacobi bound for the order of differential systems cannot be valid if a very natural conjecture concerning the differential dimension of such systems is false.

1. Summary. It will be shown that the conjectured Jacobi bound for the order of differential systems cannot be valid if a very natural conjecture concerning the differential dimension of such systems is false. It is indeed obvious that in certain cases in which the Jacobi bound is $-\infty$ (so that it is considered to hold only if the system has no component of finite order) the Jacobi bound conjecture must be false if the conjecture concerning dimension is false. It will be shown that if the conjecture concerning dimension is false, then there are even cases in which the Jacobi bound is finite but invalid.

2. Notation. K is throughout an ordinary differential field of characteristic 0. Superscripts denote differentiation.

3. Introduction. Let S be a system of k differential polynomials in $n > k$ differential indeterminates with coefficients in K . A very natural conjecture [8, Appendix, paragraph 10] is that if the manifold of S is not empty, then every component of the manifold is of positive differential dimension. This conjecture will be referred to as the *dimension conjecture*. If the dimension conjecture is false, then the strong Jacobi bound for order (definition below) must be invalid. For let S be a system which violates the dimension conjecture, and let T be obtained from S by adjoining the 0 polynomial $n - k$ times. Then T is a system of n differential polynomials in n indeterminates. The strong Jacobi bound for T is $-\infty$, but T has a component of differential dimension 0. The relation between the dimension conjecture and the conjecture that the strong Jacobi bound is valid when it is $-\infty$ has been investigated in detail by Joseph Tomasovic [9, §6] for partial as well as ordinary differential equations. He proves the equivalence of these conjectures as well as other results.

We are left with the question of the relation between the dimension conjecture and the strong Jacobi bound in the case that this bound is finite. It is the purpose of this

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note to show that if the dimension conjecture is false, then the strong Jacobi bound, and also the weak Jacobi bound, are invalid in cases in which the strong Jacobi bound is finite.

REMARK. In all that follows one can replace the set of all systems of n differential polynomials in n indeterminates by appropriate subsets (see [9]).

REMARK. One does not need characteristic 0 for the proofs of the theorems of this paper. However, the dimension conjecture is obviously false if the characteristic is positive. (The system $S = (y^p - az^p)$, where p is the characteristic and $a' \neq 0$, has only the solution $y = z = 0$.) In the spirit of the preceding remark, one might consider the dimension conjecture and Jacobi bound for systems satisfying an appropriate separability condition.

REMARK. The dimension conjecture is only known to hold for linear systems and in the case $k = 1$. It holds also for partial differential equations in these cases.

REMARK. With minor modifications the results of this paper carry over to difference polynomials.

4. Definitions and known results. Let $A = (a_{ij})$ be a $k \times n$ matrix. A *transversal* of A is a subfamily of the a_{ij} with no two entries in the same row or in the same column of A . A transversal with $\min(n, k)$ members is a *diagonal*. Let $\mathfrak{J}(A) = \max_D \sum_D a_{ij}$, where \sum_D is the sum over the diagonal D of A .

Let $S = (P_1, \dots, P_k)$ be a family of k differential polynomials in n differential indeterminates y_1, \dots, y_n . Let r_{ij} be the order of P_i in y_j , if y_j or any of its derivatives appears effectively in P_i , and let $r_{ij} = -\infty$ otherwise. Set $r_{ij}^* = \max(r_{ij}, 0)$. Let $R = (r_{ij})$, $R^* = (r_{ij}^*)$. We call R and R^* the *strong* and *weak order matrices* of S respectively.

The *strong Jacobi number* $\mathfrak{J}(S)$ of S is defined by $\mathfrak{J}(S) = \mathfrak{J}(R)$, where R is the strong order matrix of S and the obvious conventions are to be used with $-\infty$. The *weak Jacobi number* $\mathfrak{J}^*(S)$ of S is defined by $\mathfrak{J}^*(S) = \mathfrak{J}(R^*)$, where R^* is the weak order matrix of S .

It is conjectured that these numbers are bounds for the order, at least if $k = n$. More precisely, we have the following two conjectures:

If S is a system with $k = n$, and M is a component of the manifold of S with differential dimension 0, then the order of M does not exceed $\mathfrak{J}(S)$.

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The *Ritt number* $\mathfrak{R}(S)$ is defined to be the sum of the column maxima of R^* (and the definition extends in an obvious way to systems of infinitely many differential polynomials of bounded orders), and the *dual Ritt number* $\mathfrak{D}\mathfrak{R}(S)$ is the sum of the row maxima of R^* .

Let S have coefficients in the field K of characteristic 0, and let $k = n$. It is known that $\mathfrak{J}(S)$ and $\mathfrak{J}^*(S)$ are bounds for the order of the manifold of S if S is linear [7, 4, p. 199] and are the best possible bounds based on the information in R and R^* respectively [6, p. 133]. Without linearity, but with $n \leq 2$, they are bounds for the order of each component of the manifold of S of differential dimension 0 (see [7, 8, p. 136]). These results have been extended (except for being best possible) to

partial differential systems by Tomaso [9] even without the requirement $n = k$. Lando [6] has shown that $\mathfrak{J}^*(S)$ is a bound for the order of each component of the manifold of S of differential dimension 0 provided all $r_{ij} \leq 1$.

$\mathfrak{R}(S)$ is known to be a bound for the order of the components of the manifold of S of differential dimension 0 [8, p. 135]; and this is so without the assumption $n = k$ and, indeed, for infinite systems of bounded orders in the indeterminates. The work of B. Greenspan and of Lando summarized in [1] has provided improvements on the bound $\mathfrak{R}(S)$, but seems unlikely to have yielded bounds which are best possible for the information in R or R^* .

5. Theorem of König and Egerváry.

THEOREM. *Let A be a $k \times n$ matrix whose entries are 0's and 1's. Let the maximal number of 1's in any diagonal of A be $J < \min(n, k)$. Then A contains a submatrix of 0's with a rows and b columns, $0 < a, b$; $a + b = n + k - J$.*

This theorem may be obtained by an easy restatement of the result proved in [5, p. 240]; and it is evident at once from the standard form to which Barbara Lando [6, p. 129] reduces matrices of 0's and 1's.

6. The main result.

THEOREM. *If the dimension conjecture is false for systems of differential polynomials with coefficients in some extension of K , then there exists a family T of differential polynomials with coefficients in some extension of K such that T has as many polynomials as the number of differential indeterminates it involves, $\mathfrak{J}(T) \neq -\infty$, and the manifold of T has a component of differential dimension 0 and order exceeding $\mathfrak{J}(T)$. [In short, if the dimension conjecture is false in extensions of K , then the strong Jacobi number, even when finite, is not a valid bound for the order in extensions of K .]*

PROOF. By the hypothesis there exists a system $S = (P_1, \dots, P_k)$ of differential polynomials in $L\{y_1, \dots, y_n\}$, $n > k$, L an extension of K , such that some component M of the manifold of S has differential dimension 0.

Let S be chosen so that k is minimal for all choices of L . Note that $k > 1$, and so $n > 2$. Let R be the strong order matrix of S . It will be shown that R has a diagonal none of whose entries is $-\infty$. If not, it follows from the theorem of König and Egerváry that R has a submatrix H with a rows and b columns, $0 < a, b$; $a + b > n$, all of whose entries are $-\infty$. Let the numbering be such that H is in the lower right-hand corner of R . Let $c = k - a$, $d = n - b$. Note that $c \neq 0$, for otherwise all components of S have positive differential dimension. Let (a_1, \dots, a_n) be a generic zero of M . Let P_i^* result from P_i , $1 \leq i \leq c$, by the substitution of a_1, \dots, a_d for y_1, \dots, y_d . Let $S^* = (P_1^*, \dots, P_c^*)$. Suppose $b < n$. Then S^* is a system of c differential polynomials in the b indeterminates y_{d+1}, \dots, y_n with coefficients in an extension of K . Let (b_{d+1}, \dots, b_n) be a solution of S^* . Then $(a_1, \dots, a_d, b_{d+1}, \dots, b_n)$ is a solution of S . Hence, S^* has a component of differential dimension 0. But since $c < n - a < b$, this contradicts the minimality of k . If $b = n$, $P_k = 0$, and we may at

once replace S by (P_1, \dots, P_{k-1}) , obtaining a contradiction. These contradictions show that R has a diagonal with only finite entries.

It follows from the preceding paragraph that we may choose a $k \times k$ submatrix A of R with $\mathfrak{S}(A) > -\infty$. Renumber so that A consists of the first k columns of R . We may assume that M is of order exceeding $\mathfrak{S}(A)$. For if y_n is replaced in S by $y_n^{(t)}$, then S goes over into a system S' with a component M' of differential dimension 0 and order t + order of M ; while all else remains unchanged (see Appendix).

We extend S to a system S_1 by adjoining $n - k$ new indeterminates w_i and $2(n - k)$ new polynomials $A_i, B_i, k + 1 \leq i \leq n$; where for each such $i, A_i = B_i = w_i - y_i$. The system S_1 has as many polynomials as indeterminates.

Let R_1 be the strong order matrix for S_1 . Then R and A are submatrices of R_1 . For any $i, k + 1 \leq i \leq n$, the row of R_1 which corresponds to either A_i or B_i has all its entries $-\infty$, except in the columns which correspond to the $y_i, w_i, k + 1 \leq i \leq n$, in which columns it has 0 entries. Hence, a diagonal D of R_1 which is free of the entry $-\infty$ must have the entry 0 in all rows corresponding to an A_i or B_i . The diagonal D consists therefore of a diagonal of the submatrix A free of the entry $-\infty$, and of entries 0 from the rows corresponding to the A_i and B_i . Conversely, every diagonal of A free of the entry $-\infty$ extends to a diagonal of R_1 free of the entry $-\infty$ by adjoining 0 entries. It follows that $\mathfrak{S}(S_1) = \mathfrak{S}(R_1) = \mathfrak{S}(A)$. Since the manifold of S_1 evidently has a component M_1 of the same order as M , and the order of M exceeds $\mathfrak{S}(A)$, we may choose $T = S_1$.

7. The weak Jacobi bound. Let $S = (P_1, \dots, P_n)$ be a system of polynomials in $K\{y_1, \dots, y_n\}$ such that $-\infty < \mathfrak{S}(S) < \mathfrak{S}^*(S)$; and suppose also that the manifold of S has a component M of differential dimension 0 and order exceeding $\mathfrak{S}(S)$. Let R be the strong order matrix of S , and R^* the weak order matrix.

Let S_1 be formed from S by replacing each $y_i^{(K)}$ in P_j by $y_i^{(K+1)}, 1 \leq i, j \leq n, K = 0, 1, \dots$. Let R_1, R_1^* be respectively the strong and weak order matrices of S_1 .

The entries of R_1^* exceed the corresponding entries of R^* by 1, except that those 0 entries in R^* which correspond to the entries $-\infty$ in R remain as 0 in R_1^* . Because $\mathfrak{S}(S) < \mathfrak{S}^*(S)$, the sum $\mathfrak{S}^*(S)$ is obtained only from diagonals of R^* which correspond to diagonals of R with at least one entry $-\infty$. These diagonals of R^* give rise to diagonals of R_1^* with sum at most $\mathfrak{S}^*(S) + n - 1$. All other diagonals of R have entries with sum less than $\mathfrak{S}^*(S)$, and therefore correspond to diagonals of R_1 whose entries have sum less than $\mathfrak{S}^*(S) + n$. It follows that $\mathfrak{S}^*(S_1) = \mathfrak{S}(R_1^*) < \mathfrak{S}^*(S) + n$. On the other hand $\mathfrak{S}(S)$ is the sum of the entries in a diagonal of R free of the entry $-\infty$; and this diagonal gives rise to a diagonal of R_1 whose entries have sum $\mathfrak{S}(S) + n$, which is clearly the maximal sum for diagonals of R_1 . Hence, $\mathfrak{S}(S_1) = \mathfrak{S}(R_1) = \mathfrak{S}(S) + n$. It follows that $0 \leq \mathfrak{S}^*(S_1) - \mathfrak{S}(S_1) < \mathfrak{S}^*(S) - \mathfrak{S}(S)$.

To the component M of the manifold of S there corresponds a component M_1 of the manifold of S_1 of order n + order of M (see Appendix). After a sufficient number h of repetitions of the above procedure one obtains a system S_h such that $\mathfrak{S}^*(S_h) = \mathfrak{S}(S_h) = \mathfrak{S}(S) + nh$. The manifold of S_h has a component M_h of order nh + order M . Hence, the order of M_h exceeds $\mathfrak{S}(S) + nh = \mathfrak{S}^*(S_h)$. We have thus obtained the following corollary to the theorem of the preceding section.

COROLLARY 1. *Let K be an ordinary differential field. If the dimension conjecture is false for systems of differential polynomials with coefficients in some extension of K , then the weak Jacobi number is not a bound for the order of components of systems with finite strong Jacobi number and coefficients in some extension of K .*

A similar result can be proved for the dual Ritt number. Let $S = (P_1, \dots, P_n)$ denote a family of differential polynomials in $K\{y_1, \dots, y_n\}$ such that $\mathfrak{S}(S)$ is finite and S has a component M of differential dimension 0 and order exceeding $\mathfrak{S}^*(S)$. Let R^* be the weak order matrix for S . By a result well known in connection with the solution of the assignment problem of integer programming, but actually due to Jacobi [2], there exist nonnegative integers d_1, \dots, d_n such that the matrix R' obtained by adding d_i to each entry in the i th column of R^* , $1 \leq i \leq n$, has a diagonal consisting of row maxima. Then $\mathfrak{D}\mathfrak{R}(R') = \mathfrak{S}^*(R')$. It follows as in the earlier discussions that the dual Ritt number is not a bound for the orders of the components of differential dimension 0 of the system S' obtained from S by replacing y_i by $y_i^{(d_i)}$, $1 \leq i \leq n$. This proves the following corollary.

COROLLARY 2. *Let K be an ordinary differential field. If the dimension conjecture is false for systems of differential polynomials with coefficients in extensions of K , then the dual Ritt number is not a bound for the order of components of systems with finite strong Jacobi number and coefficients in some extension of K .*

REMARK. It is not evident how one could apply the method used to prove Corollary 2 to obtain a similar result for the Ritt number (which would amount to a proof of the dimension conjecture and the Jacobi bounds, since the Ritt number is a valid bound). For to do so one would have to modify a system S not by replacing the indeterminates by their derivatives of appropriate orders, but by replacing the polynomials of the system by their derivatives of appropriate orders. The effect of such a replacement on the order of the system is not apparent.

Appendix. The following lemma justifies the substitutions used in the preceding proofs.

LEMMA. *Let K be an ordinary differential field. Let the manifold of a system S of differential polynomials of $K\{y_1, \dots, y_n\}$ have a component M of differential dimension 0. Let S' be obtained from S by replacing $y_1^{(K)}$ by $y_1^{(K+1)}$, $K = 0, 1, \dots$, in all the polynomials of S . Then the manifold of S' has a component M' of order $1 + \text{order of } M$.*

PROOF. Let (a_1, \dots, a_n) be a generic zero of M . Let b be a generic zero of the prime differential ideal $I = \{z - a_1\}$ in $K\langle a_1, \dots, a_n \rangle\{z\}$. Then (b, a_2, \dots, a_n) is a solution of S' , and

$$\begin{aligned} \text{t.d. } K\langle b; a_2, \dots, a_n \rangle / K &= 1 + \text{t.d. } K\langle a_1, \dots, a_n \rangle / K \\ &= 1 + \text{order of } M. \end{aligned}$$

Let M' be a component of S' admitting the solution (b, a_2, \dots, a_n) . Let (g_1, \dots, g_n) be a generic zero of M' . Then $(g'_1; g_2, \dots, g_n)$ is a zero of S specializing differentially over K to (a_1, \dots, a_n) . This specialization must be generic since M is a component of

the manifold of S . Now $\text{t.d. } K\langle g_1, \dots, g_n \rangle / K \geq \text{t.d. } K\langle b, a_2, \dots, a_n \rangle / K = 1 + \text{order of } M$. But, $\text{t.d. } K\langle g_1, \dots, g_n \rangle / K \leq 1 + \text{t.d. } K\langle g'_1; g_2, \dots, g_n \rangle / K = 1 + \text{order of } M$; where the equality holds because $(g'_1; g_2, \dots, g_n)$ is generic for M over K . Thus, $\text{ord } M' = \text{t.d. } K\langle g_1, \dots, g_n \rangle / K = 1 + \text{order of } M$.

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