

ON PRODUCTIVE CLASSES OF FUNCTION RINGS

PAUL BANKSTON

ABSTRACT. No nontrivial P -class (“ P ” for “productive”) of rings of continuous real-valued functions can be category equivalent to any elementary P -class of finitary universal algebras.

0. Introduction. In this paper, “algebra” means “finitary universal algebra” in the sense of Birkhoff, and a class \mathfrak{K} of algebras will be viewed as a category by allowing all algebra homomorphisms as category morphisms. \mathfrak{K} is *productive* or a *P -class* (resp. *S -class*) if \mathfrak{K} is closed under usual direct products (resp. subalgebras); \mathfrak{K} is *elementary* if there is a set of first order axioms such that \mathfrak{K} is precisely the class of models of those axioms (see [5]).

We will be interested in category theoretic properties of classes of function rings, to wit: Let RCF denote the class of all rings of continuous real-valued functions $C(X)$ with topological spaces for domains. We ask which subclasses of RCF can be equivalent to “nice” classes of algebras (e.g. elementary P -classes, SP -classes, varieties, etc.).

0.1 **EXAMPLES.** (i) $\mathfrak{K}_0 = \{C(X) : X \text{ is zero-dimensional compact Hausdorff}\}$ is equivalent to the variety of Boolean algebras (see [6]).

(ii) $\mathfrak{K}_1 = \{C(X) \in \mathfrak{K}_0 : X \text{ has no isolated points}\}$ is equivalent to the elementary P -class of atomless Boolean algebras.

There has come to be a growing list of negative results in this area. In [1] it is shown that $\mathfrak{K} = \{C(X) : X \text{ is compact Hausdorff}\}$ cannot be equivalent to an SP -class; and in [3] it is shown also that \mathfrak{K} cannot be equivalent to a class \mathfrak{L} which is “representable” (i.e. free objects over singletons exist in \mathfrak{L}) and is either an elementary P -class or an S -class whose basic alphabet of operation symbols has cardinality less than that of the continuum. The major unsolved problem in this area is whether \mathfrak{K} is equivalent to any elementary P -class at all. In this paper we highlight the importance of the fact that products in \mathfrak{K}_0 , \mathfrak{K}_1 and \mathfrak{K} above are not the usual ones by proving results of which the following is an easy corollary.

0.2 **THEOREM.** *No nontrivial P -subclass of RCF (i.e. one having more than the isomorphism type of the “degenerate” ring $0 = C(\emptyset)$) can be category equivalent to an elementary P -class.*

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The proof employs the notion of “reduced product” in a category and will be presented in the next section.

1. Main results. We assume the reader to be familiar with the usual notion of reduced product in model theory [5]. The key observation, one which is well known, is that if \mathcal{K} is any elementary P -class of finitary relational structures, then ultraproducts in \mathcal{K} are simply direct limits of direct products (see [2, 3, 4] for more details and references). This of course can be placed in a categorical context. We will show that most categorical reduced products in a P -subclass of RCF must be trivial (or nonexistent). This will immediately entail (0.2) since the diagonal morphism from an algebra into an ultrapower in an elementary P -class is always a monomorphism.

Our notation regarding reduced products and powers comes from [5]: If $\langle A_i; i \in I \rangle$ is a family of relational structures of the same finitary type and D is a filter on I then $\prod_D A_i = \prod_D \langle A_i; i \in I \rangle$ is the reduced product with elements $a_D = \{a' \in \prod_{i \in I} A_i; \{i: a'_i = a_i\} \in D\}$; if each A_i is equal to A then the reduced power is denoted $\prod_D A$ and the natural diagonal embedding is denoted $d: A \rightarrow \prod_D A$.

For $J \supseteq K \in D$, let $r_{JK}: \prod_{i \in J} A_i \rightarrow \prod_{i \in K} A_i$ be the natural restriction morphism. Then the associated direct limit $\varinjlim \langle \prod_{i \in J} A_i; J \in D \rangle$ is precisely the reduced product $\prod_D A_i$ in the category of all relational structures. Moreover, if \mathcal{K} is any elementary P -class then categorical ultraproducts in \mathcal{K} are the usual ones. (N.B. It is possible to have an elementary class \mathcal{K} which has unusual ultraproducts as a category (see [4]).)

Before we state our main results, we introduce the notion of “commuting system” of homomorphisms. Let $\langle X_i; i \in I \rangle$ be topological spaces, let D be a filter on I , and let X be any space. A “commuting system”, in this context, is a family $\langle h_J; J \in D \rangle$ where, for $J \in D$, $h_J: \prod_{i \in J} C(X_i) \rightarrow C(X)$ is a homomorphism such that for all $J \supseteq K \in D$, $h_K \circ r_{JK} = h_J$. Our main concern is in the existence of certain commuting systems.

1.1 THEOREM. *Assume $\langle X_i; i \in I \rangle$ is a family of topological spaces, D is a free filter on I (i.e. $\bigcap D = \emptyset$), X is a space, there are no uncountable measurable cardinals at most $|I|$ (= the cardinality of I), and $\langle h_J; J \in D \rangle$ is a commuting system. Then X is empty.*

PROOF. It is well known (see [6]) that we lose no generality by assuming all of the above spaces to be realcompact Tichonov; and we can then invoke Theorem (10.6) of [6] to the effect that if Y is realcompact and if $h: C(Y) \rightarrow C(X)$ is a ring homomorphism (N.B. $h(1) = 1$. Hence $\text{Hom}(0, C(X)) = \emptyset$, unless $X = \emptyset$.) then there is a unique continuous $h': X \rightarrow Y$ such that, for all $f \in C(Y)$, $h(f) = f \circ h'$.

Let $\dot{\bigcup}_{i \in I} X_i$ denote the disjoint union of the spaces X_i ; and let p be a z -ultrafilter on $\dot{\bigcup}_{i \in I} X_i$ with the countable intersection property (c.i.p. = intersections of countable subfamilies of p are nonempty). We show that $\dot{\bigcup}_{i \in I} X_i$ is realcompact by proving that p must be fixed. Indeed let $g: \dot{\bigcup}_{i \in I} X_i \rightarrow I$ take x to i exactly when $x \in X_i$, and let $F = \{J \subseteq I: g^{-1}[J] \in p\}$. One can check easily enough that F is a countably complete ultrafilter on I (e.g. F is closed under superset since $g^{-1}[J] = \dot{\bigcup}_{i \in J} X_i$ is always clopen, hence a zero set). Now there are no uncountable measurable cardinals

at most $|I|$, hence F must be fixed (= principal). Suppose $\{i\} \in F$. Then $X_i \in p$, and p restricted to X_i is a z -ultrafilter on X_i with the c.i.p. Thus, since X_i is realcompact, p converges.

Now since $\prod_{i \in I} C(X_i)$ and $C(\dot{\bigcup}_{i \in I} X_i)$ are naturally isomorphic, we can consider each h_j as a homomorphism from $C(\dot{\bigcup}_{i \in J} X_i)$ to $C(X)$. Thus look at the “dual system” $h'_j: X \rightarrow \dot{\bigcup}_{i \in J} X_i$. Letting $e_{JK}: \dot{\bigcup}_{i \in K} X_i \rightarrow \dot{\bigcup}_{i \in J} X_i$ be the natural embedding, $J \supseteq K \in D$ (an inclusion in this case), we note that the uniqueness of each h'_j ensures that all the appropriate diagrams commute (i.e. $e_{JK} \circ h'_K = h'_J$ for each $J \supseteq K \in D$). Since $\bigcap D = \emptyset$, this forces X to be empty. \square

1.2 COROLLARY. *If \mathcal{K} is a P -subclass of RCF then reduced products in \mathcal{K} are “trivial”, in the sense that $\prod_D^{\mathcal{K}} A_i$, the reduced product in \mathcal{K} , is zero whenever D is a free filter on an index set whose cardinality is less than all uncountable measurable cardinals.* \square

1.3 REMARK. The measurable cardinal hypothesis is necessary for 1.1 to work. For let D be a free countably complete ultrafilter on a set I , and let each X_i be a singleton. Then $\prod_D C(X_i) \cong \mathbf{R}$ (= the ring of real numbers), by Corollary (4.2.8) in [5].

We can get the conclusion of 1.1 with altered hypotheses and more model theoretic techniques.

1.4 THEOREM. *Assume $\langle X_i: i \in I \rangle$ is a family of spaces, D is a countably incomplete ultrafilter on I , X is a space, and $\langle h_j: J \in D \rangle$ is a commuting system. Then X is empty.*

PROOF. Suppose, to the contrary, that there is a nonempty space X for which a commuting system exists. If $\{i: X_i = \emptyset\} = J \in D$ then $h_j: \prod_{i \in J} C(X_i) \rightarrow C(X)$, being a ring homomorphism, forces X to be empty. Since D is an ultrafilter, we lose no generality by assuming that $X_i \neq \emptyset$ for each $i \in I$. For each $J \in D$ let $p_J: \prod_{i \in J} C(X_i) \rightarrow \prod_D C(X_i)$ be the natural projection homomorphism. By properties of direct limits there is a unique homomorphism $h: \prod_D C(X_i) \rightarrow C(X)$ such that for all $J \in D$, $h \circ p_J = h_j$. Now for each $i \in I$ let $d_i: \mathbf{R} \rightarrow C(X_i)$ be the diagonal embedding. Then the ultraproduct mapping $\prod_D d_i: \prod_D \mathbf{R} \rightarrow \prod_D C(X_i)$ is a homomorphism. Now D is an ultrafilter, hence $\prod_D \mathbf{R}$ is a field by the Łoś Theorem. Therefore $h \circ \prod_D d_i$ is a homomorphism from a field into a nontrivial ring, hence an embedding. Let $d: \mathbf{R} \rightarrow \prod_D \mathbf{R}$, $e: \mathbf{R} \rightarrow C(X)$ be the diagonal embeddings. It is a straightforward algebraic fact that there can be no other homomorphism $e': \mathbf{R} \rightarrow C(X)$, since $C(X)$ is a “diagonal” subring of a power of \mathbf{R} (use the fact that the identity map is the only ring endomorphism on \mathbf{R}). Therefore $e = h \circ \prod_D d_i \circ d$, and $\prod_D \mathbf{R}$ embeds as a diagonal subring of $C(X)$. Since D is countably incomplete, this ultrapower is ω_1 -saturated. We will obtain a contradiction once we prove the

LEMMA. *No diagonal subring of a power of \mathbf{R} is ω_1 -saturated.*

PROOF OF LEMMA. Let $A \subseteq \mathbf{R}^I$ be a diagonal subring which is ω_1 -saturated. For each $n \in \omega$, let $\phi_n(x)$ be the first order formula which says of x that $x - n$ (= the

result when the constantly n function is subtracted from the function x) has a square root. $\phi_n(x)$ can be expressed in the first order language of rings with countably many additional constants. Clearly $\Phi(x) = \{\phi_n(x) : n \in \omega\}$ is finitely satisfiable in A : if $\Phi_0(x) = \{\phi_{n_1}(x), \dots, \phi_{n_k}(x)\}$ then $A \models \phi_{n_i}[\max\{n_1, \dots, n_k\}]$ for $i = 1, \dots, k$ since $\mathbf{R} \subseteq A$. So by ω_1 -saturicity, there is an $a \in A$ such that $a - n$ has a square root for each $n \in \omega$. That is, for each $i \in I$, the i th coordinate a_i of a exceeds n for all $n \in \omega$, a contradiction. \square

1.5 COROLLARY. *If \mathcal{K} is a P -subclass of RCF then ultraproducts in \mathcal{K} are "trivial", in the sense that $\prod_D^{\mathcal{K}} A_i$ is zero whenever D is a countably incomplete ultrafilter.* \square

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DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE, MARQUETTE UNIVERSITY, MILWAUKEE, WISCONSIN 53233