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COHOMOLOGY OF HEISENBERG LIE ALGEBRAS

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ABSTRACT. The cohomology of Heisenberg Lie algebras is studied and we obtain the description of cocycles, coboundaries and cohomological spaces.

1. Notations and preliminaries.

1.1. Let $\underline{g}$ be a Lie algebra of dimension n over a field F and M a $\underline{g}$-module of finite dimension over F. We denote by $C^p(\underline{g}, M)$ the space of cochains of degree p, d_p: $C^p(\underline{g}, M) \to C^{p+1}(\underline{g}, M)$ the restriction to $C^p(\underline{g}, M)$ of the coboundary operator, $Z^p(\underline{g}, M)$ the kernel of d_p (space of cocycles of degree p), $B^p(\underline{g}, M)$ the range of d_{p-1} (space of coboundaries of degree p), $H^p(\underline{g}, M)$ the quotient of $Z^p(\underline{g}, M)$ by $B^p(\underline{g}, M)$ (space of cohomology of degree p of $\underline{g}$ with values in M). If $M = F$, denote $C^p(\underline{g}) = C^p(\underline{g}, F)$, $Z^p(\underline{g}) = Z^p(\underline{g}, F)$, $B^p(\underline{g}) = B^p(\underline{g}, F)$, $H^p(\underline{g}) = H^p(\underline{g}, F)$. For all details see [**1, 2, 5, 6**].

1.2. By the vector space isomorphisms

$$C^p(\underline{g}, M) \cong M \otimes_F \bigwedge^p \underline{g}^*, \qquad C^p(\underline{g}, M)/Z^p(\underline{g}, M) \cong B^{p+1}(\underline{g}, M)$$

(where $\underline{g}^*$ is the dual of $\underline{g}$ and $\bigwedge^p \underline{g}^*$ the vector space of homogeneous elements of degree p of the Grassmann algebra over $\underline{g}^*$) one has

$$\binom{n}{p} \dim M = \dim Z^p(\underline{g}, M) + \dim B^{p+1}(\underline{g}, M);$$

therefore,

(i) $\dim H^p(\underline{g}, M) = \dim Z^p(\underline{g}, M) + \dim Z^{p-1}(\underline{g}, M) - \binom{n}{p-1} \dim M$,

(ii) $\dim H^p(\underline{g}, M) = \binom{n}{p} \dim M - \dim B^p(\underline{g}, M) - \dim B^{p+1}(\underline{g}, M)$.

Denote

$$\chi_p(\underline{g}, M) = \sum_{q=0}^{p} (-1)^q \dim H^q(\underline{g}, M)$$

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(partial sum of the Euler-Poincaré characteristic); by using $\sum_{q=0}^{p}(-1)^q\binom{n}{q-1} = (-1)^p\binom{n-1}{p-1}$ one obtains

(iii) $$\dim Z^p(\underline{g}, M) = (-1)^p\chi_p(\underline{g}, M) + \binom{n-1}{p-1}\dim M,$$

(iv) $$\dim B^{p+1}(\underline{g}, M) = (-1)^p\chi_p(\underline{g}, M) + \binom{n-1}{p}\dim M.$$

(REMARK. Since $C^{n+1}(\underline{g}, M) = (0)$ one has $\dim B^{n+1}(\underline{g}, M) = 0$, therefore (iv) gives for the case $p = n$: $0 = (-1)^n\chi_n(\underline{g}, M) + 0\cdot\dim M$; thus $\chi_n(\underline{g}, M) = 0$. Goldberg obtains this result for $M = F$ [**3**].)

1.3. Let $\underline{g}$ and $\underline{g}'$ be two Lie algebras of dimension n and n' over F, ρ and ρ' two representations of $\underline{g}$ and $\underline{g}'$ into a vector space M, ϕ: $\underline{g} \to \underline{g}'$ a Lie algebra morphism such that $\rho = \rho' \circ \phi$. Denote

$$\phi_p = {}^t\left(\bigwedge^p \phi\right) = \bigwedge^p {}^t\phi\colon C^p(\underline{g}', M) \to C^p(\underline{g}, M);$$

then, obviously, $\phi_{p+1} \circ d'_p = d_p \circ \phi_p$; since $\operatorname{Im}\phi_p = \bigwedge^p(\operatorname{Ker}\phi)^\perp$ it follows that

$$\phi_p(Z^p(\underline{g}', M)) \subset \bigwedge^p(\ker\phi)^\perp \cap Z^p(\underline{g}, M),$$

$$\phi_p(B^p(\underline{g}', M)) \subset \bigwedge^p(\ker\phi)^\perp \cap B^p(\underline{g}, M).$$

1.4. LEMMA. *With the above notations, if ϕ is onto then*

(i) $$\phi_p(Z^p(\underline{g}', M)) = \bigwedge^p(\ker\phi)^\perp \cap Z^p(\underline{g}, M),$$

(ii) $$\dim H^p(\underline{g}', M) \leqslant \dim H^p(\underline{g}, M) + \left(\binom{n}{p-1} - \binom{n'}{p-1}\right)\dim M,$$

(iii) $$\dim H^p(\underline{g}, M) \leqslant \dim H^p(\underline{g}', M) + \left(\binom{n}{p} - \binom{n'}{p}\right)\dim M.$$

PROOF. If ϕ is onto then ϕ_p is one-to-one; let $f \in \operatorname{Im}\phi_p \cap Z^p(\underline{g}, M)$. One can write $f = \phi_p f'$ with $f' \in C^p(\underline{g}', M)$; then $0 = d_p f = d_p\phi_p f' = \phi_{p+1}d'_p f'$; therefore $d'_p f' = 0$, i.e. $f' \in Z^p(\underline{g}', M)$, which proves (i).

Since ϕ_p is one-to-one, we have

$$\dim Z^p(\underline{g}', M) = \dim\phi_p(Z^p(\underline{g}', M)) \leqslant \dim Z^p(\underline{g}, M);$$

thus

$$\dim H^p(\underline{g}', M) \leqslant \dim Z^p(\underline{g}, M) + \dim Z^{p-1}(\underline{g}, M) - \binom{n'}{p-1}\dim M$$

(by 1.2(i)), which proves (ii) (by 1.2(i) again).

By considering B^p instead of Z^p, (iii) is obtained in the same way.

1.5. REMARKS. (i) The inclusion $\phi_p(B^p(\underline{g}', M)) \subset \bigwedge^p(\ker\phi)^\perp \cap B^p(\underline{g}, M)$ may be strict even with ϕ onto. For example let $\underline{g} = \mathfrak{H}_m$ be the Heisenberg Lie algebra of dimension $2m + 1 \geqslant 5$ (2.2), $\underline{g}' = Fe'_1 \oplus \cdots \oplus Fe^*_{2m}$ the abelian Lie algebra of dimension $2m$ and ϕ: $\mathfrak{H}_m \to \underline{g}'$, $e_i \mapsto e'_i$, $i \neq 0$, $e_0 \mapsto 0$. One has (by (2.2))

$$B^p(\mathfrak{H}_m, M) \subset Z^p(\mathfrak{H}_m, M) = \bigwedge^p (e^{*1}, \dots, e^{*2m}) = \bigwedge^p (Fe_0)^{\perp} = \bigwedge^p (\ker \phi)^{\perp};$$

thus

$$B^p(\mathfrak{H}_m, M) \cap \bigwedge^p (\ker \phi)^{\perp} = B^p(\mathfrak{H}_m, M);$$

since $\phi_p(B^p(\underline{g}', M)) = \phi_p((0)) = (0)$ and $B^p(\mathfrak{H}_m, M) \cong \wedge^p(e^{*1}, \dots, e^{*2m})$ (by (2.2)), the inclusion cannot be an equality.

(ii) For the above example of $\mathfrak{H}_m$ and $\underline{g}'$, 1.4(ii) is an equality (by (2.2)). In low dimension there are several other examples realizing the equality and proving thus that the inequality cannot be improved.

2. Cohomological spaces of Heisenberg Lie algebras.

2.1. Remark. By the Poincaré duality $H^p(\mathfrak{H}_m)$ and $H^{2m+1-p}(\mathfrak{H}_m)$ are canonically isomorphic; therefore, one has to study only $H^p(\mathfrak{H}_m)$ for $p \leqslant m$.

2.2. Theorem. *Let $\mathfrak{H}_m = Fe_0 \oplus \cdots \oplus Fe_{2m}$ be the Heisenberg Lie algebra of dimension $2m + 1$ over a commutative field F, i.e. a Lie algebra satisfying $[e_i, e_{i+m}] = e_0$ $\forall i = 1, \dots, m$ (all the other brackets are 0). Let $p \in \{0, \dots, m\}$ and denote by $\{e^{*0}, \dots, e^{*2m}\}$ the dual basis of $\{e_0, \dots, e_{2m}\}$. Then*

(i) *the pth Betti number (i.e.* $\dim H^p(\mathfrak{H}_m)$) *is equal to* $\binom{2m}{p} - \binom{2m}{p-2}$;

(ii) *the space of cocycles of degree p of $\mathfrak{H}_m$ with values in F is equal to the vector space of homogeneous elements of degree p of the Grassmann algebra over $(e^{*1}, \dots, e^{*2m})$, i.e.*

$$Z^p(\mathfrak{H}_m) = \bigwedge^p (e^{*1}, \dots, e^{*2m});$$

(iii) *the space of coboundaries of degree p of $\mathfrak{H}_m$ with values in F is isomorphic to the vector space of homogeneous elements of degree $p - 2$ of the Grassmann algebra over $(e^{*1}, \dots, e^{*2m})$, the isomorphism being given by the exterior product by $d_1 e^{*0}$:*

$$\bigwedge^{p-2} (e^{*1}, \dots, e^{*2m}) \tilde{\to} B^p(\mathfrak{H}_m), \qquad \gamma \mapsto d_1 e^{*0} \wedge \gamma.$$

Proof. We will use induction on $m \geqslant 1$. The case $m = 1$ is obvious.

Let $\underline{g}' = Fe'_1 \oplus \cdots \oplus Fe'_{2m}$ be the abelian Lie algebra of dimension $2m$ and ϕ: $\mathfrak{H}_m \to \underline{g}'$ the morphism defined by $\phi e_i = e'_i$ $\forall i = 1, \dots, 2m$, $\phi e_0 = 0$. In the notation of 1.3 one takes $M = F$, $\rho = \rho' = 0$; then, by 1.4(ii),

$$\binom{2m}{p} \leqslant \dim H^p(\mathfrak{H}_m) + \binom{2m+1}{p-1} - \binom{2m}{p-1}$$

and therefore $\binom{2m}{p} - \binom{2m}{p-2} \leqslant \dim H^p(\mathfrak{H}_m)$.

Let $\mathfrak{H} = Fe_0 \oplus \cdots \oplus Fe_{2m-1}$ and $\mathfrak{H}_{m-1} = Fe_0 \oplus \cdots \oplus \widehat{Fe_m} \oplus \cdots \oplus Fe_{2m-1}$ be the ideals of $\mathfrak{H}_m$ such that $\mathfrak{H} = \mathfrak{H}_{m-1} \times Fe_m$ (direct product). By the theorem of

Künneth [6, Chapter II, §5, p. 80] one has

$$H^p(\mathfrak{H}_{m-1} \times Fe_m) \cong \sum_{i+j=p} H^i(\mathfrak{H}_{m-1}) \otimes H^j(Fe_m);$$

therefore $\dim H^p(\mathfrak{H}) = \dim H^p(\mathfrak{H}_{m-1}) + \dim H^{p-1}(\mathfrak{H}_{m-1})$. Since $\mathfrak{H}_{m-1}$ is a Heisenberg Lie algebra of dimension $2(m-1)+1$, the induction hypothesis gives $\dim H^p(\mathfrak{H}_{m-1}) = \binom{2m-2}{p} - \binom{2m-2}{p-2}$ for $p \leq m-1$ and thus $\dim H^p(\mathfrak{H}) = \binom{2m-1}{p} - \binom{2m-1}{p-2}$ for $p \leq m-1$. By Proposition 1 of [2] one has the following exact sequence (because $\mathfrak{H}$ is an ideal of codimension 1 in $\mathfrak{H}_m$):

$$\cdots \to H^{p-1}(\mathfrak{H}) \to H^p(\mathfrak{H}_m) \to H^p(\mathfrak{H}) \to H^p(\mathfrak{H}) \to H^{p+1}(\mathfrak{H}_m) \to \cdots$$

which is a special case of the Hochschild-Serre spectral sequence [5]. Therefore, $\dim H^p(\mathfrak{H}_m) \leq \dim H^p(\mathfrak{H}) + \dim H^{p-1}(\mathfrak{H})$ and thus $\dim H^p(\mathfrak{H}_m) \leq \binom{2m}{p} - \binom{2m}{p-2}$ for $p \leq m-1$. Since the opposite inequality has already been proved one has conclusion (i) for $p \leq m-1$.

For $p = m$, one has to look more closely at the exact sequence of Dixmier. We proved that for $p \leq m-1$,

$$\dim H^p(\mathfrak{H}_m) = \dim H^p(\mathfrak{H}) + \dim H^{p-1}(\mathfrak{H});$$

therefore, the long exact sequence splits into small exact sequences

$$\begin{gathered}
(0) \to H^0(\mathfrak{H}_m) \to H^0(\mathfrak{H}) \to (0), \\
(0) \to H^0(\mathfrak{H}) \to H^1(\mathfrak{H}_m) \to H^1(\mathfrak{H}) \to (0), \\
(0) \to H^1(\mathfrak{H}) \to H^2(\mathfrak{H}_m) \to H^2(\mathfrak{H}) \to (0), \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
(0) \to H^{m-2}(\mathfrak{H}) \to H^{m-1}(\mathfrak{H}_m) \to H^{m-1}(\mathfrak{H}) \to (0), \\
(0) \to H^{m-1}(\mathfrak{H}) \to H^m(\mathfrak{H}_m) \to H^m(\mathfrak{H}) \xrightarrow{u} H^m(\mathfrak{H}).
\end{gathered}$$

One then has $\dim H^m(\mathfrak{H}_m) = \dim H^{m-1}(\mathfrak{H}) + \dim \operatorname{Ker} u$. Let $\alpha \in H^m(\mathfrak{H}_{m-1})$ and $\beta \in H^{m-1}(\mathfrak{H}_{m-1})$; one has

$$\alpha \wedge \beta = c(\alpha, \beta) e^{*0} \wedge \cdots \wedge \hat{e}^{*m} \wedge \cdots \wedge e^{*2m-1} \quad \text{with } c(\alpha,\beta) \in F.$$

The duality of Poincaré asserts that

$$H^m(\mathfrak{H}_{m-1}) \times H^{m-1}(\mathfrak{H}_{m-1}) \to F, \qquad (\alpha, \beta) \to c(\alpha, \beta)$$

is a nondegenerate bilinear form; therefore $H^m(\mathfrak{H}_{m-1})$ is a quotient of

$$\bigwedge^m \left(e^{*0} \cdots \hat{e}^{*m} \cdots e^{*2m-1}\right)$$

(since $H^{m-1}(\mathfrak{H}_{m-1})$ is a quotient of $\bigwedge^{m-1}(e^{*1} \cdots \hat{e}^{*m} \cdots e^{*2m-1})$). On the other hand, by the theorem of Künneth,

$$H^m(\mathfrak{H}) = H^m(\mathfrak{H}_{m-1}) + H^{m-1}(\mathfrak{H}_{m-1}) \wedge e^{*m}.$$

Now u is the action of e_{2m} on $H^m(\mathfrak{H})$ [2, Proposition 1]; since $[e_m e_{2m}] = e_0$, $[e_i e_{2m}] = 0$, $i \neq m$, one has $e_{2m} \cdot e^{*0} = e^{*m}$, $e_{2m} \cdot e^{*i} = 0$, $i \neq m$; therefore

$$\operatorname{Im} u = \operatorname{Ker} u = H^{m-1}(\mathfrak{H}_{m-1}) \wedge e^{*m};$$

thus

$$\dim \operatorname{Ker} u = \dim H^{m-1}(\mathfrak{H}_{m-1}) = \binom{2m-2}{m-1} - \binom{2m-2}{m-3}$$

(induction). We then have

$$\begin{aligned}\dim H^m(\mathfrak{H}_m) &= \binom{2m-1}{m-1} - \binom{2m-1}{m-3} + \binom{2m-2}{m-1} - \binom{2m-2}{m-3} \\ &= \binom{2m}{m} - \binom{2m}{m-2}.\end{aligned}$$

This proves conclusion (i) for $p = m$.

By a simple computation one has $(-1)^p \chi_p(\mathfrak{H}_m, F) = \binom{2m}{p} - \binom{2m}{p-1}$ and by 1.2(iii), another computation gives $\dim Z^p(\mathfrak{H}_m) = \binom{2m}{p}$. From 1.3 it follows that $\phi_p(Z^p(g')) \subset Z^p(\mathfrak{H}_m)$. Since $Z^p(g') = \bigwedge^p(e'^{*1},\dots,e'^{*2m})$, we get $\phi_p(Z^p(g')) = \bigwedge^p(e^{*1},\dots,e^{*2m})$, therefore $\bigwedge^p(e^{*1},\dots,e^{*2m}) \subset Z^p(\mathfrak{H}_m)$. Conclusion (ii) then follows from $\dim Z^p(\mathfrak{H}_m) = \binom{2m}{p}$.

Now

$$B^p(\mathfrak{H}_m) = \left\{ d_{p-1}\alpha;\ \alpha \in \bigwedge^{p-1}(e^{*0},\dots,e^{*2m}) \right\}.$$

Any $\alpha \in \bigwedge^{p-1}(e^{*0},\dots,e^{*2m})$ can be written $\alpha = \beta + e^{*0} \wedge \gamma$ with $\beta \in \bigwedge^{p-1}(e^{*1},\dots,e^{*2m})$ $(= Z^{p-1}(\mathfrak{H}_m)$ by (ii)) and $\gamma \in \bigwedge^{p-2}(e^{*1},\dots,e^{*2m})$ $(= Z^{p-2}(\mathfrak{H}_m)$ by (ii)); thus $d_{p-1}\alpha = 0 + d_1 e^{*0} \wedge \gamma - e^{*0} \wedge 0$ (recall that d_p is an antiderivation); therefore

$$B^p(\mathfrak{H}_m) = \left\{ d_1 e^{*0} \wedge \gamma;\ \gamma \in \bigwedge^{p-2}(e^{*1},\dots,e^{*2m}) \right\}.$$

Let $\psi\colon \bigwedge^{p-2}(e^{*1},\dots,e^{*2m}) \to B^p(\mathfrak{H}_m)$, $\gamma \mapsto d_1 e^{*0} \wedge \gamma$; by the above result, ψ is onto. By (i) and (ii), $\dim B^p(\mathfrak{H}_m) = \binom{2m}{p-2}$; thus $\dim B^p(\mathfrak{H}_m) = \dim \bigwedge^{p-2}(e^{*1},\dots,e^{*2m})$, which proves that ψ is an isomorphism.

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