RELATIVE NORMAL COMPLEMENTS IN FINITE GROUPS

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Abstract. \((G, H, H_0, \pi)\) denotes the following configuration: \(H\) and \(H_0\) are the subgroups of the finite group \(G\) with \(H_0 \leq H\) and \(\pi\) is the set of primes dividing \((H : H_0)\). For \((G, H, H_0, \pi)\) we consider conditions (A), (B), and (C): (A) Any two \(\pi\)-elements of \(H - H_0\) which are \(G\)-conjugate are \(H\)-conjugate. (B) For each \(\pi\)-element \(x \in H - H_0\), \(C_G(x) = I(x)C_H(x)\) where \(I(x)\) is a normal \(\pi\)-subgroup of \(C_G(x)\). (C) \(|(H - H_0)^{G,\pi}| = (G : H) |H - H_0|\). We show that if \((G, H, H_0, \pi)\) satisfies (B) and (C), or (A) and (B), and if \(H/H_0\) is solvable, then there is a unique relative normal complement \(G_0\) of \(H\) over \(H_0\).

All groups in this paper are finite. Given a group \(G\) with subgroups \(H_0, H,\) and \(G_0\) such that \(H_0 \leq H\), we call \(G_0\) a relative normal complement of \(H\) over \(H_0\) if \(G_0 \leq G, G = G_0 H\) and \(H_0 = G_0 \cap H\).

Let \(\pi(G)\) denote the set of primes dividing \(|G|\). If \(\pi\) is a set of primes, then the complementary set of primes will be denoted by \(\pi'\). A group \(G\) is a \(\pi\)-group if \(\pi(G) \subseteq \pi\). If \(x \in G\), then \(x\) is a \(\pi\)-element if \(|x|\) is a \(\pi\)-group. Every element \(x\) of \(G\) has a unique decomposition \(x = x_\pi x_{\pi'} = x_\pi x_{\pi'}\) into a \(\pi\)-element \(x_\pi\) and a \(\pi'\)-element \(x_{\pi'}\). Further, \(x_\pi\) and \(x_{\pi'}\) are powers of \(x\). If \(x\) and \(y\) are elements of a subgroup \(K\) of \(G\), then \(x\) and \(y\) belong to the same \(\pi\)-section of \(K\) if their \(\pi\)-parts \(x_\pi\) and \(y_\pi\) are \(K\)-conjugate. If \(S\) is a subset of \(G\), then \(S^{G,\pi}\) denotes the union of all \(\pi\)-sections of \(G\) which intersect \(S\).

We let \((G, H, H_0, \pi)\) denote the following configuration: \(G\) is a finite group with subgroups \(H\) and \(H_0\) such that \(H_0 \leq H\) and \(\pi = \pi(H/H_0)\). For \((G, H, H_0, \pi)\) we consider the following conditions:

(A) Any two \(\pi\)-elements of \(H - H_0\) which are \(G\)-conjugate are \(H\)-conjugate.
(B) For each \(\pi\)-element \(x \in H - H_0\), \(C_G(x) = I(x)C_H(x)\) where \(I(x)\) is a normal \(\pi\)-subgroup of \(C_G(x)\).
(C) \(|(H - H_0)^{G,\pi}| = (G : H) |H - H_0|\).

Leonard [2] has shown that if \((G, H, H_0, \pi)\) satisfies conditions (B) and (C) and \(\pi = \{p\}\) or \(I(x)\) is always a Hall \(\pi\)-subgroup of \(C_G(x)\), then there is a unique relative normal complement \(G_0\) of \(H\) over \(H_0\) and \(G_0 = G - (H - H_0)^{G,\pi}\). If \((G, H, H_0, \pi)\) satisfies conditions (A) and (B) and \(\pi = \{p\}\), then Reynolds [3] has shown that there is a unique relative normal complement \(G_0\) of \(H\) over \(H_0\) and \(G_0 = G - (H - H_0)^{G,\pi}\). In this paper, we prove two generalizations of these theorems.
Theorem 1. If \((G, H, H_0, \pi)\) satisfies conditions \((B_0)\) and \((C)\) and \(H/H_0\) is solvable, then there is a unique relative normal complement \(G_0\) of \(H\) over \(H_0\) and \(G_0 = G - (H - H_0)^G_\pi\).

Theorem 2. If \((G, H, H_0, \pi)\) satisfies conditions \((A)\) and \((B_0)\) and \(H/H_0\) is solvable, then there is a unique relative normal complement \(G_0\) of \(H\) over \(H_0\) and \(G_0 = G - (H - H_0)^G_\pi\).

We omit stating explicitly the obvious corollaries which follow from Theorems 1 and 2 by replacing "\(H/H_0\) is solvable" by "\(\pi\) is a set of odd primes".

If a set \(S\) is a disjoint union of \(S\) and \(T\), we write \(R = S \cup T\).

Lemma 1. Assume \((G, H, H_0, \pi)\) satisfies condition \((B_0)\), \(K\) is a subgroup of \(H\) containing \(H_0\) and \(\pi_2 \subseteq \pi\). Then

\[
(K - H_0)^G_\pi : (H - K)^G_\pi \subseteq (H - H_0)^G_\pi.
\]

Proof. Assume \(w \in (H - K)^G_\pi : (H - H_0)^G_\pi\), then \(w = w_\pi w_{\pi_2}\) and \(w_\pi = w_\pi w_{\pi_2}\), where \(\pi_2 = \pi - \pi_2\). Further, \(w_{\pi_2} = x^g\) where \(x\) is a \(\pi_2\)-element in \(H - K\) or \(x\) is a \(\pi_2\)-element in \(K - H_0\). Since \(K \supseteq H_0\), \(x\) is a \(\pi\)-element in \(H - H_0\); hence \(w_\pi = (w_{\pi_2})^{-1}\) and \(w = h_\pi x = h_\pi x_\pi\), where \(x_\pi \in C_G(x) = (H - K)^G_\pi\). Let \(\tilde{w} = (w_{\pi_2})^{-1}\), then \(\tilde{w}l(x) = h_\pi x\) where \(h_\pi \in C_{H_\pi}(x)\). Since \(l(x)\) is a normal \(\pi\)-subgroup of \(C_G(x)\) and \(\tilde{w}\) is a \(\pi\)-element, \(h_\pi\) may be taken to be a \(\pi\)-element of \(H\). Now \(h_\pi x = h_\pi x\) and \(\tilde{w}l(x) = h_\pi x\) and Theorem 6.2.1 of [1] imply that \(h_\pi = h_\pi x\) where \(t \in I(x)\). It follows that \(h_\pi x = \tilde{w}\) for some integer \(i\), and \(h_\pi x = \tilde{w}_{\pi_2} = x\). Since \(t \in C_{H_\pi}(x)\), \(h_\pi\) is a \(\pi\)-element of \(H - H_0\) and \(\tilde{w}_{\pi_2}\) is conjugate to an element of \(H - H_0\). Thus, \(w \in (H - H_0)^G_\pi\).

Proof of Theorem 1. If \((G, H, H_0, \pi)\) satisfies conditions \((B_0)\) and \((C)\), then Leonard showed in §3 of [2], that \(H - H_0\) is a union of \(\pi\)-sections of \(H\) and condition \((A)\) is satisfied.

Let \(G\) be a minimal counterexample to Theorem 1. Theorem 1.2 of [2] implies that \(\pi\) contains more than one prime. Since \(H/H_0\) is solvable, there is a prime \(p \in \pi\) such that \(H\) has an abelian \(p\) factor group. Thus, \(H\) contains a normal subgroup \(H_1\) of index \(p\) and \(H_1 \supset H_0\). Let \(D = H - H_1\), \(x \in D\) and \(y \in H\) where \(x_p\) and \(y_p\) are \(H\)-conjugate. Then \(x_p\) and \(y_p\) \(\in H_1\) and \(x_p \in H - H_1\). Thus, \(y_p \in H - H_1\) and \(y \in H - H_1\). Therefore, \(D\) is a union of \(\pi\)-sections of \(H\). If \(x\) and \(y\) are \(p\)-elements in \(D\) which are \(G\)-conjugate, then \(x\) and \(y\) are \(G\)-conjugate \(\pi\)-elements in \(H - H_0\). Condition \((A)\) implies \(x\) and \(y\) are \(H\)-conjugate. If \(x\) is a \(p\)-element in \(D\), then \(C_G(x) = O_{p'}(C_G(x))C_{H_1}(x)\) follows from condition \((B_0)\). Applying Theorem 2 of [3] to \(G, H, H_1\), we see that \(G\) has a unique relative normal complement \(G_1\) of \(H\) over \(H_1\) and \(G_1 = G - (H - H_1)^{G_1}_{\pi}G_1\). Now \(G = G_1 \cup (H - H_1)^{G_1}_{\pi}\) and \((G_1 : H_1) = (G : H)\) yield

\[
|(H - H_1)^{G_1}_{\pi}| = |G| - |G_1| = |G_1| \cdot ((H : H_1) - 1) = (G_1 : H_1) |H - H_1|.
\]

Let \(R = G_1 \cap (H - H_0)^{G_\pi}\) and \(\pi_1 = \pi(H_1/H_0)\). We will show that \((G_1, H_1, H_0, \pi_1)\) satisfies the hypothesis of Theorem 1 and \(R = (H_1 - H_0)^{G_1}_{\pi_1}\). It follows from Lemma 1, \(\pi_1 \subseteq \pi\) and \(H_1 \supset H_0\), that \((H_1 - H_0)^{G_1}_{\pi_1} \subseteq R\). Let \(w \in R\), then \(w_\pi = x^g\) where \(x\) is a \(\pi\)-element in \(H - H_0\). We may write \(g = hy\) where \(h \in H\).
and \( y \in G_1 \). Let \( \tilde{w} = w^x \) and \( \tilde{x} = x^h \), then \( \tilde{w} \in G_1 \); hence, \( \tilde{x} \in G_1 \cap (H - H_0) = H_1 - H_0 \). Since \( y \in G_1 \), it follows that \( w \in (H_1 - H_0)^{G_1, \pi} \) if \( \pi_1 = \pi \). If \( \pi \neq \pi_1 \), then \( \tilde{x} = \tilde{x}_{\pi_1} \tilde{x}_p \). Now \( H_1/H_0 \) a \( p' \)-group implies that \( \tilde{x}_p \in H_0 \). Thus, \( \tilde{x}_{\pi_1} \in H_1 - H_0 \) and \( \tilde{x}_{\pi_1} = \tilde{x}_{\pi_1} \). Again \( y \in G_1 \) so \( w \in (H_1 - H_0)^{G_1, \pi} \). Therefore, \( R = (H_1 - H_0)^{G_1, \pi} \).

Lemma 1, applied to \( H_1 \) and \( p \), implies that \((H - H_0)^{G_1, \pi} \subseteq (H - H_0)^{G_1, \pi} \). Using \( G = G_1 \cup (H - H_1)^{G_1, \pi} \), we see that \((H - H_0)^{G_1, \pi} = R \cup (H - H_1)^{G_1, \pi} \). Now condition (C) and \((G : H) = (G_1 : H_1) \) yield

\[
\left| (H_1 - H_0)^{G_1, \pi} \right| = R = \left| (H - H_0)^{G_1, \pi} \right| - \left| (H - H_1)^{G_1, \pi} \right|
\]

\[
= (G_1 : H_1)\left( |H - H_0| - |H - H_1| \right) = (G_1 : H_1)\left| H_1 - H_0 \right|.
\]

Hence, \((G_1, H_1, H_0, \pi) \) satisfies condition (C).

Let \( x \) be a \( \pi_1 \)-element in \( H_1 - H_0 \). Then \( x \) is a \( \pi \)-element in \( H - H_0 \); hence, \( C_{G_1}(x) = I(x)C_{H_1}(x) \). Since \( \pi_1 \subseteq \pi \) and \((G : G_1) = p \), \( I(x) \subseteq G_1 \) and \( I(x) \) is a normal \( \pi_1 \)-subgroup of \( C_{G_1}(x) \). Further, \( C_{G_1}(x) = I(x)C_{H_1}(x) \cap G_1 = I(x)(C_{H_1}(x) \cap G_1) = I(x)C_{H_1}(x) \) so that condition (B) is satisfied by \((G_1, H_1, H_0, \pi_1) \).

Clearly \( H_1/H_0 \) is solvable since \( H/H_0 \) is. The minimality of \( |G| \) now implies that \( G_1 \) has a unique relative normal complement \( G_0 \) of \( H_1 \) over \( H_0 \), namely \( G_0 = G_1 - (H_1 - H_0)^{G_1, \pi_1} \). It has been shown that \((H_1 - H_0)^{G_1, \pi_1} = R \) is a normal subset of \( G \).

Thus, \( G_0 \) is a normal subgroup of \( G \). Clearly, \( G_0 \) is a relative normal complement of \( H \) over \( H_0 \). Proposition 2.2 of [2] now implies that \( G_0 \) is unique and \( G = G - (H - H_0)^{G, \pi} \).

Proof of Theorem 2. Assume \((G, H, H_0, \pi) \) satisfies the hypothesis of Theorem 2 and let \( D = H - H_0 \). Assume \( x \in D \) and \( y \in H \) where \( x_\pi \) is \( H \)-conjugate to \( y_\pi \). Then \( x_\pi \) and \( y_\pi \in H_0 \) and \( x_\pi \in H - H_0 \). Thus, \( y_\pi \in H - H_0 \); hence, \( y \in H - H_0 \) and \( D \) is a union of \( \pi \)-sections of \( H \).

If \( S \) is a \( \pi \)-section of \( G \) and \( S \cap D \neq \emptyset \), let \( x \) and \( y \in S \cap D \). Then \( x \) and \( y \in H - H_0 \), in particular, \( x_\pi \) and \( y_\pi \in H - H_0 \) with \( x_\pi \) and \( y_\pi \) \( G \)-conjugate. Using condition (A), we see that \( x_\pi \) and \( y_\pi \) are \( H \)-conjugate; hence, \( x \) and \( y \) lie in the same \( \pi \)-section of \( H \). Therefore, \( S \cap D = \emptyset \) or \( S \cap D \) is a \( \pi \)-section of \( H \). Lemma 3.4 of [2] now implies that

\[
\left| (H - H_0)^{G, \pi} \right| = |D^{G, \pi}| = (G : H) \left| D \right| = (G : H) \left| H - H_0 \right|.
\]

Thus, \((G, H, H_0, \pi) \) satisfies conditions (B) and (C), and \( H/H_0 \) is solvable. Theorem 2 follows from Theorem 1.

References