PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 1, January 1983

RELATIVE NORMAL COMPLEMENTS IN FINITE GROUPS

PAMELA A. FERGUSON

ABSTRACT. (G, H, H_0, π) denotes the following configuration: H and H_0 are the subgroups of the finite group G with $H_0 \trianglelefteq H$ and π is the set of primes dividing $(H : H_0)$. For (G, H, H_0, π) we consider conditions (A), (B_0), and (C): (A) Any two π-elements of $H - H_0$ which are G-conjugate are H-conjugate. (B_0) For each π-element $x \in H - H_0$, $C_G(x) = I(x)C_H(x)$ where $I(x)$ is a normal π'-subgroup of $C_G(x)$. (C) $|(H - H_0)^{G,\pi}| = (G : H)|H - H_0|$. We show that if (G, H, H_0, π) satisfies (B_0) and (C), or (A) and (B_0), and if H/H_0 is solvable, then there is a unique relative normal complement G_0 of H over H_0.

All groups in this paper are finite. Given a group G with subgroups H_0, H, and G_0 such that $H_0 \trianglelefteq H$, we call G_0 a relative normal complement of H over H_0 if $G_0 \trianglelefteq G$, $G = G_0H$ and $H_0 = G_0 \cap H$.

Let $\pi(G)$ denote the set of primes dividing $|G|$. If π is a set of primes, then the complementary set of primes will be denoted by π'. A group G is a π-group if $\pi(G) \subseteq \pi$. If $x \in G$, then x is a π-element if $\langle x \rangle$ is a π-group. Every element x of G has a unique decomposition $x = x_\pi x_{\pi'} = x_{\pi'} x_\pi$ into a π-element x_π and a π'-element $x_{\pi'}$. Further, x_π and $x_{\pi'}$ are powers of x. If x and y are elements of a subgroup K of G, then x and y belong to the same π-section of K if their π-parts x_π and y_π are K-conjugate. If S is a subset of G, then $S^{G,\pi}$ denotes the union of all π-sections of G which intersect S.

We let (G, H, H_0, π) denote the following configuration: G is a finite group with subgroups H and H_0 such that $H_0 \trianglelefteq H$ and $\pi = \pi(H/H_0)$. For (G, H, H_0, π) we consider the following conditions:

(A) Any two π-elements of $H - H_0$ which are G-conjugate are H-conjugate.

(B_0) For each π-element $x \in H - H_0$, $C_G(x) = I(x)C_H(x)$ where $I(x)$ is a normal π'-subgroup of $C_G(x)$.

(C) $|(H - H_0)^{G,\pi}| = (G : H)|H - H_0|$.

Leonard [2] has shown that if (G, H, H_0, π) satisfies conditions (B_0) and (C) and $\pi = \{p\}$ or $I(x)$ is always a Hall π'-subgroup of $C_G(x)$, then there is a unique relative normal complement G_0 of H over H_0 and $G_0 = G - (H - H_0)^{G,\pi}$. If (G, H, H_0, π) satisfies conditions (A) and (B_0) and $\pi = \{p\}$, then Reynolds [3] has shown that there is a unique relative normal complement G_0 of H over H_0 and $G_0 = G - (H - H_0)^{G,\pi}$. In this paper, we prove two generalizations of these theorems.

Received by the editors January 13, 1982 and, in revised form, June 9, 1982.
1980 *Mathematics Subject Classification*. Primary 20E07.

0002-9939/82/0000-0703/$01.75

THEOREM 1. *If (G, H, H_0, π) satisfies conditions* (B_0) *and* (C) *and H/H_0 is solvable, then there is a unique relative normal complement G_0 of H over H_0 and* $G_0 = G - (H - H_0)^{G,\pi}$.

THEOREM 2. *If (G, H, H_0, π) satisfies conditions* (A) *and* (B_0) *and H/H_0 is solvable, then there is a unique relative normal complement G_0 of H over H_0 and* $G_0 = G - (H - H_0)^{G,\pi}$.

We omit stating explicitly the obvious corollaries which follow from Theorems 1 and 2 by replacing "H/H_0 is solvable" by "π is a set of odd primes".

If a set R is a disjoint union of S and T, we write $R = S \dot{\cup} T$.

LEMMA 1. *Assume (G, H, H_0, π) satisfies condition* (B_0), *K is a subgroup of H containing H_0 and $\pi_2 \subseteq \pi$. Then*

$$(K - H_0)^{G,\pi_2} \cup (H - K)^{G,\pi_2} \subseteq (H - H_0)^{G,\pi}.$$

PROOF. Assume $w \in (H - K)^{G,\pi_2} \cup (K - H_0)^{G,\pi_2}$, then $w = w_\pi w_{\pi'}$ and $w_\pi = w_{\pi_2} w_{\pi_3}$ where $\pi_3 = \pi - \pi_2$. Further, $w_{\pi_2} = x^g$ where x is a π_2-element in $H - K$ or x is a π_2-element in $K - H_0$. Since $K \supseteq H_0$, x is a π-element in $H - H_0$; hence $w^{g^{-1}} \in C_G(x) = I(x)C_H(x)$. Let $\tilde{w} = (w_\pi)^{g^{-1}}$, then $\tilde{w}I(x) = hI(x)$ where $h \in C_H(x)$. Since $I(x)$ is a normal π'-subgroup of $C_G(x)$ and $\tilde{w}$ is a π-element, h may be taken to be a π-element of H. Now $\langle \tilde{w} \rangle I(x) = \langle h \rangle I(x)$ and Theorem 6.2.1 of [1] imply that $\langle \tilde{w} \rangle = \langle h \rangle^t$ where $t \in I(x)$. It follows that $h^{it} = \tilde{w}$ for some integer i, and $h^{it}_{\pi_2} = \tilde{w}_{\pi_2} = x$. Since $t \in C_G(x)$, h^t is a π-element of $H - H_0$ and $\tilde{w}_\pi$ is conjugate to an element of $H - H_0$. Thus, $w \in (H - H_0)^{G,\pi}$.

PROOF OF THEOREM 1. If (G, H, H_0, π) satisfies conditions (B_0) and (C), then Leonard showed in §3 of [2], that $H - H_0$ is a union of π-sections of H and condition (A) is satisfied.

Let G be a minimal counterexample to Theorem 1. Theorem 1.2 of [2] implies that π contains more than one prime. Since H/H_0 is solvable, there is a prime $p \in \pi$ such that H has an abelian p factor group. Thus, H contains a normal subgroup H_1 of index p and $H_1 \supset H_0$. Let $D = H - H_1$, $x \in D$ and $y \in H$ where x_p and y_p are H-conjugate. Then $x_{p'}$ and $y_{p'} \in H_1$ and $x_p \in H - H_1$. Thus, $y_p \in H - H_1$ and $y \in H - H_1$. Therefore, D is a union of p-sections of H. If x and y are p-elements in D which are G-conjugate, then x and y are G-conjugate π-elements in $H - H_0$. Condition (A) implies x and y are H-conjugate. If x is a p-element in D, then $C_G(x) = O_{p'}(C_G(x))C_H(x)$ follows from condition (B_0). Applying Theorem 2 of [3] to G, H, and H_1, we see that G has a unique relative normal complement G_1 of H over H_1 and $G_1 = G - (H - H_1)^{G,p}$. Now $G = G_1 \dot{\cup} (H - H_1)^{G,p}$ and $(G_1 : H_1) = (G : H)$ yield

$$|(H - H_1)^{G,p}| = |G| - |G_1| = |G_1|((H : H_1) - 1) = (G_1 : H_1)|H - H_1|.$$

Let $R = G_1 \cap (H - H_0)^{G,\pi}$ and $\pi_1 = \pi(H_1/H_0)$. We will show that (G_1, H_1, H_0, π_1) satisfies the hypothesis of Theorem 1 and $R = (H_1 - H_0)^{G_1,\pi_1}$. It follows from Lemma 1, $\pi_1 \subseteq \pi$ and $H_1 \supseteq H_0$, that $(H_1 - H_0)^{G_1,\pi_1} \subseteq R$. Let $w \in R$, then $w_\pi = x^g$ where x is a π-element in $H - H_0$. We may write $g = hy$ where $h \in H$

and $y \in G_1$. Let $\tilde{w} = w_\pi^{y^{-1}}$ and $\tilde{x} = x^h$, then $\tilde{w} \in G_1$; hence, $\tilde{x} \in G_1 \cap (H - H_0) = H_1 - H_0$. Since $y \in G_1$, it follows that $w \in (H_1 - H_0)^{G_1, \pi_1}$ if $\pi_1 = \pi$. If $\pi \neq \pi_1$, then $\tilde{x} = \tilde{x}_{\pi_1}\tilde{x}_p$. Now H_1/H_0 a p'-group implies that $\tilde{x}_p \in H_0$. Thus, $\tilde{x}_{\pi_1} \in H_1 - H_0$ and $\tilde{w}_{\pi_1} = \tilde{x}_{\pi_1}$. Again $y \in G_1$ so $w \in (H_1 - H_0)^{G_1, \pi_1}$. Therefore, $R = (H_1 - H_0)^{G_1, \pi_1}$. Lemma 1, applied to H_1 and p, implies that $(H - H_1)^{G,p} \subseteq (H - H_0)^{G,\pi}$. Using $G = G_1 \,\dot{\cup}\, (H - H_1)^{G,p}$, we see that $(H - H_0)^{G,\pi} = R \,\dot{\cup}\, (H - H_1)^{G,p}$. Now condition (C) and $(G : H) = (G_1 : H_1)$ yield

$$|(H_1 - H_0)^{G_1, \pi_1}| = |R| = |(H - H_0)^{G,\pi}| - |(H - H_1)^{G,p}|$$
$$= (G_1 : H_1)(|H - H_0| - |H - H_1|) = (G_1 : H_1)\,|H_1 - H_0|\,.$$

Hence, (G_1, H_1, H_0, π) satisfies condition (C).

Let x be a π_1-element in $H_1 - H_0$. Then x is a π-element in $H - H_0$; hence, $C_G(x) = I(x)C_H(x)$. Since $\pi_1 \subseteq \pi$ and $(G : G_1) = p$, $I(x) \subseteq G_1$ and $I(x)$ is a normal π_1'-subgroup of $C_G(x)$. Further, $C_{G_1}(x) = I(x)C_H(x) \cap G_1 = I(x)(C_H(x) \cap G_1) = I(x)C_{H_1}(x)$ so that condition (B_0) is satisfied by (G_1, H_1, H_0, π_1).

Clearly H_1/H_0 is solvable since H/H_0 is. The minimality of $|G|$ now implies that G_1 has a unique relative normal complement G_0 of H_1 over H_0, namely $G_0 = G_1 - (H_1 - H_0)^{G_1, \pi_1}$. It has been shown that $(H_1 - H_0)^{G_1, \pi_1} = R$ is a normal subset of G. Thus, G_0 is a normal subgroup of G. Clearly, G_0 is a relative normal complement of H over H_0. Proposition 2.2 of **[2]** now implies that G_0 is unique and $G_0 = G - (H - H_0)^{G,\pi}$.

Proof of Theorem 2. Assume (G, H, H_0, π) satisfies the hypothesis of Theorem 2 and let $D = H - H_0$. Assume $x \in D$ and $y \in H$ where x_π is H-conjugate to y_π. Then $x_{\pi'}$ and $y_{\pi'} \in H_0$ and $x_\pi \in H - H_0$. Thus, $y_\pi \in H - H_0$; hence, $y \in H - H_0$ and D is a union of π-sections of H.

If S is a π-section of G and $S \cap D \neq \varnothing$, let x and $y \in S \cap D$. Then x and $y \in H - H_0$, and in particular, x_π and $y_\pi \in H - H_0$ with x_π and y_π G-conjugate. Using condition (A), we see that x_π and y_π are H-conjugate; hence, x and y lie in the same π-section of H. Therefore, $S \cap D = \varnothing$ or $S \cap D$ is a π-section of H. Lemma 3.4 of **[2]** now implies that

$$|(H - H_0)^{G,\pi}| = |D^{G,\pi}| = (G : H)\,|D| = (G : H)\,|H - H_0|\,.$$

Thus, (G, H, H_0, π) satisfies conditions (B_0) and (C), and H/H_0 is solvable. Theorem 2 follows from Theorem 1.

References

1. D. Gorenstein, *Finite groups*, Harper & Row, New York, 1968.
2. H. Leonard, *On relative normal complements in finite groups* (to appear).
3. W. F. Reynolds, *Isometries and principal blocks of group characters*, Math. Z. **107** (1968), 264–270.

Department of Mathematics, University of Miami, Coral Gables, Florida 33124