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ON THE DISTRIBUTION OF PRIME ELEMENTS IN POLYNOMIAL KRULL DOMAINS

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ABSTRACT. Let A be a Krull domain having infinitely many height one primes. It is shown that any ideal of height two in the polynomial ring $A[t]$ contains a prime element. An application to the construction of Dedekind domains with specified class groups is given, along with an example to show the necessity of assuming infinitely many height one primes.

Introduction. Krull domains can be viewed as generalizations of factorial rings and, inversely, factorial rings can be characterized as Krull domains with trivial divisor class group. As is well known, an integral domain is factorial if and only if every one of its nonzero prime ideals contains a nonzero principal prime ideal, i.e. a prime element. Thus, to assert the factoriality of an integral domain is to make a statement about the distribution of its prime elements. Factorial rings are those in which the prime elements are in some sense widely distributed.

The purpose of this note is to present a result on the distribution of prime elements in integral domains of the form $B = A[t]$, where A is a Krull domain and t is an indeterminate. We shall prove that if A has infinitely many height one prime ideals, then every ideal in B of grade at least two contains a prime element. This theorem provides, for polynomial rings over Krull domains, a converse to the obvious fact that in any integral domain an ideal which properly contains a principal prime ideal has grade at least two. We shall give an example to show the necessity of the assumption that A has infinitely many height one prime ideals. Note that this assumption is equivalent to the assertion that A is not a semilocal principal ideal domain.

Results.

THEOREM. *Let A be a Krull domain which has infinitely many height one prime ideals, and let $B = A[t]$, where t is an indeterminate. Then any ideal of B of grade at least two (or height at least two, since B is a Krull domain) contains a principal prime ideal.*

PROOF. Let J be an ideal of B of grade at least two. Then there are elements $f(t)$, $g(t)$ in J which are a regular sequence in B, i.e. $(f(t), g(t))^{-1} = B$.

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Let K be the fraction field of A. Since $K[t]$ is a flat extension of B, $[(f(t), g(t))K[t]]^{-1} = K[t]$, and hence $(f(t), g(t))K[t] = K[t]$. This yields $(f(t), g(t))B \cap A \neq 0$.

Let $d \in (f(t), g(t))B \cap A$, $d \neq 0$. Write $f(t) = f_0 + \cdots + f_m t^m$, $g(t) = g_0 + \cdots + g_n t^n$, with each f_i and g_j in A. At least one of f_0, g_0 is nonzero, say $f_0 \neq 0$. Let $P_1, \ldots, P_u$ be the height one primes of A containing f_0 and $P'_1, \ldots, P'_v$ those containing d. Let P be any other height one prime of A. By the approximation theorem for Krull domains (or the prime avoidance lemma) we may choose an element

$$p \in P \setminus \left(P^{(2)} \cup P_1 \cup \cdots \cup P_u \cup P'_1 \cup \cdots \cup P'_v\right).$$

Here $P^{(2)}$ denotes the second symbolic power of P.

Let $h(t) = pf(t) + pg(t)t^{m+1} + dt^{m+n+2}$. Since f_0, $d \notin P$, we can apply the Eisenstein criterion to $h(t)$ considered as an element of $A_P[t]$ to conclude that $h(t)$ is irreducible in $K[t]$.

The content of $h(t)$ is the ideal $c(h)$ of A generated by the coefficients of h. We have $c(h) = pc(f) + pc(g) + dA$. Evidently, no height one prime ideal Q of A contains $c(h)$. For otherwise it would contain d, hence not p, and therefore $Q[t]$ would contain both $f(t)$ and $g(t)$, which is impossible. Thus we have $c(h)^{-1} = A$.

By [**2**, Lemma 1], $h(t)K[t] \cap B = h(t) \cdot c(h)^{-1} \cdot B = h(t)B$. But $h(t)$ is irreducible in $K[t]$, so $h(t)K[t]$ is a prime ideal, and $h(t)$ is prime in B. By our construction, $h(t) \in J$. This completes the proof.

As an application of the theorem, we now give a new (and in a certain sense simplified) proof of [**1**, Theorem 14.2]. This result is the first step in the construction of Dedekind domains with specified class groups, in that it reduces the problem to the construction of Krull domains with specified class groups.

COROLLARY. *Let A be a Krull domain. Then there is a flat extension B of A such that B is a Dedekind domain*, $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ *is an isomorphism, and B is an inert extension of A in the sense of Samuel* (*see* [**1**, *Theorem* 14.2]).

PROOF. If A is a Dedekind domain we set $B = A$, so assume that A is not a Dedekind domain. Then A has infinitely many height one prime ideals, so the theorem applies. For each prime ideal Q of $A[t]$ of height at least two, let p_Q be a prime element of $A[t]$ contained in Q. Let S be the multiplicative system in $A[t]$ generated by these p_Q. Let $B = S^{-1}A[t]$. B is a Krull domain and by our construction of S it is one-dimensional, hence is a Dedekind domain. B is clearly a flat extension of A. It is well known that $\mathrm{Cl}(A) \to \mathrm{Cl}(A[t])$ is an isomorphism. By Nagata's theorem [**1**, Corollary 7.3], $\mathrm{Cl}(A[t]) \to \mathrm{Cl}(B)$ is also an isomorphism. Finally, we see as in the proof of [**1**, Theorem 14.2] that B is an inert extension of A.

We close with an example to show the necessity of the assumption of infinitely many height one primes in the theorem.

EXAMPLE. A Krull domain A such that in $A[t]$ there exist height two ideals which contain no height one prime ideals.

Let A be a complete discrete valuation ring, and let p be the generator of its maximal ideal. Let $h(t) = (t + p)^2(t + 1) = t^3 + (1 + 2p)t^2 + (2p + p^2)t + p^2$.

Consider the ideal $J = (h(t), p^2) \subseteq A[t]$. Since A is a Noetherian factorial ring, and p^2 and $h(t)$ are relatively prime, J has height two and grade two. To see that J contains no height one prime ideal, we need only show that it contains no prime element.

Suppose that $f(t) \in J$ is a prime element, and write $f(t) = p^2 g_1(t) + h(t)g_2(t)$. If $g_2(t) \equiv 0 \pmod p$, then $f(t)$ is divisible by p. Since $f(t)$ is prime we may, in this case, assume $f(t) = p$. Then $p = p^2 g_1(t) + h(t)g_2(t)$ leads to the absurdity $p \in p^2 A$, upon comparing constant terms. Thus we may suppose that $g_2(t) \not\equiv 0 \pmod p$. Using bars to denote images modulo p, we have $0 \neq \bar{f}(t) = \bar{h}(t)\bar{g}_2(t) = t^2(t+1)\bar{g}_2(t)$. Thus we may factor $\bar{f}(t)$ as $\bar{f}(t) = t^r \bar{g}(t)$, where t and $\bar{g}(t)$ are relatively prime and $r \geqslant 2$. Since $t+1$ divides $\bar{g}(t)$, deg $\bar{f}(t) > r$, and hence deg $f(t) > r$. By Hensel's lemma, $f(t) = l(t)g(t)$, where $l(t) \equiv t^r \pmod p$ and $l(t)$ is monic of degree r. It follows that $f(t)$ is not irreducible in $A[t]$, contradicting the assumption that it is prime. This shows that J contains no prime element.

References

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